Determination of Ice Stream Bed Strength via the Incomplete Gauss-Newton Method

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The Pretend Shallow Shelf Approximation

$$
\begin{aligned}
-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \gamma \, \frac{u}{\sqrt{u^2+v^2}} &= f_x \\
-\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} + \gamma \, \frac{v}{\sqrt{u^2+v^2}} &= f_y
\end{aligned} \tag{1}
$$

Forward problem: Given a bed strength coefficient γ, find ice velocities (u, v) solving (1). I.e. $(u, v) = \mathcal{F}_{\mathrm{SSA}}(\gamma)$.

Inverse problem: Given observed ice velocities (u, v), determine the corresponding bed strength coefficient γ.

Inverse problem is ill-posed

Problem 1: (u, v) has twice as many degrees of freedom as γ. We can't expect any solution at all to exist.

Problem 2: The inverse map $\mathcal{F}_{\mathrm{SSA}}^{-1}$ is not continuous. An estimate for the amount of error in (u, v) does not imply an estimate for the amount of error in γ.

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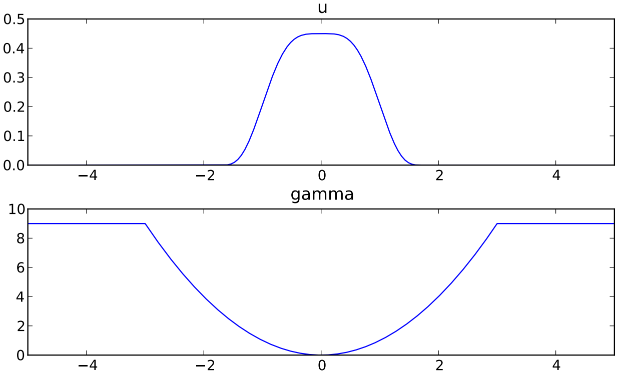
These problems have been addressed in the ice literature by minimizing

$$J(\gamma) = \int_\Omega |(u, v) - \mathcal{F}_{\mathrm{SSA}}(\gamma)|^2,$$

typically with steepest descent.

The forward map is smoothing



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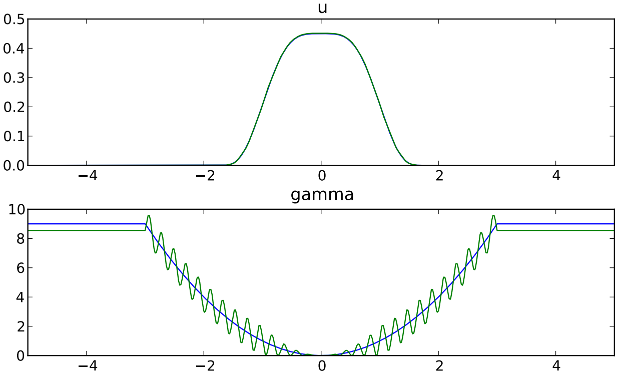

The inverse problem I'd really love to solve

Among all parameters γ, find the 'least featured' one such that $\mathcal{F}_{\mathrm{SSA}}(\gamma)$ is 'consistent' with observation.

For example, minimize

$$||\gamma||_X$$

subject to

$$||\mathcal{F}_{\mathrm{SSA}}(\gamma) - (u, v)||_Y < \delta$$

where δ is specified in advance and incorporates estimates for both model and measurement error.

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Standard approaches to regularization

Tikhonov Regularization:

Minimize

$$J_\nu(\gamma) = J(\gamma) + \nu||\gamma - \gamma_0||_X$$

where ν is a regularization parameter (TBD).

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Iterative Methods:

- Start with an initial estimate γ_0 for the bed strength and a desired misfit level δ.
- Iteratively determine a sequence of search directions $\{h_k\}$.
- Determine $\gamma_{k+1} = \gamma_k + t_k h_k$ with t_k minimizing $t \mapsto J(\gamma_k + t h_k)$.
- Stop at the first iteration k such that $J(\gamma_k) < \delta$.

Why you might believe iterative methods work

For steepest descent,

$$h_k = T^*((u_k, v_k) - (u, v)),$$

where $(u_k, v_k) = \mathcal{F}_{\mathrm{SSA}}(\gamma_k)$ and $T = \mathcal{F}'_{\mathrm{SSA}}(\gamma_k)$.

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- Because $\mathcal{F}_{\mathrm{SSA}}$ smooths wiggles, so does T.
- Because T smooths wiggles, so does T^*.
- So the search direction is a 'smoothed out' version of the residual.
- Other minimization methods based on the gradient (e.g. nonlinear conjugate gradient method) can be expected to have this same property.

Gauss-Newton Method

Method for minimizing nonlinear least-squares problem, e.g.

$$J(\gamma) = \|y - \mathcal{F}_{\mathrm{SSA}}(\gamma)\|_Y^2.$$

- At iterate γ_k, define

$$F_k(h) = \mathcal{F}_{\mathrm{SSA}}(\gamma_k) + \mathcal{F}'_{\mathrm{SSA}}(x)[h] \approx \mathcal{F}_{\mathrm{SSA}}(x_k + h)$$

- Determine a search direction h_k by minimizing the quadratic functional

$$J_k(h) = \|y - F_k(h)\|_Y^2$$

- Determine $\gamma_{k+1} = \gamma_k + t_k h_k$ with t_k minimizing $t \mapsto J(\gamma_k + t h_k)$.

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But if the original minimization problem for J is ill-posed, so is the minimization problem for J_k.

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Iteratively Regularized Gauss-Newton:

Determine search directions h_k by minimizing

$$J_k(h) = \|y - F_k(h)\|_Y^2 + \nu_k \|h\|_X^2$$

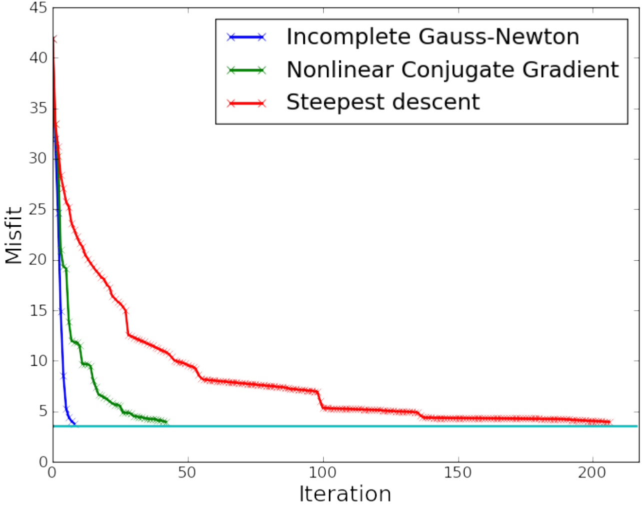
Incomplete Gauss Newton

- Start with an initial estimate γ_0 for the bed strength and a desired misfit level δ.
- At iteration γ_k, determine the current misfit δ_k, and construct quadratic functional J_k.
- Use linear conjugate gradient method on J_k to correct a fraction θ of the remaining misfit $\delta_k - \delta$.
- This determines a search direction h_k, now minimize $t \mapsto J(\gamma_k + th_k)$.
- Manage θ if the resulting misfit decrease is poor.

What's good

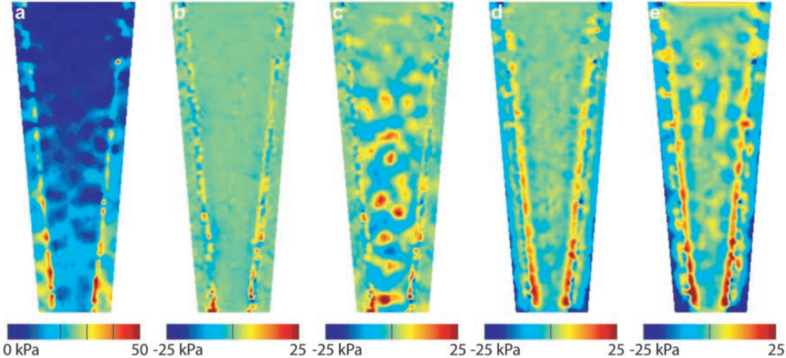
- The extra information required for regularization is very tangible: an initial estimate γ_0 and an estimate for the error δ in the measurements and model.
- The method is comparatively fast:
  - Nothing is ever optimized completely.
  - Linesearches (i.e. determining step t_k along h_k) often terminate after one nonlinear function evaluation.
  - Linear inverse problems at each iteration are cheap to solve (linear conjugate gradient method, stopping early).

What's good


Incomplete Gauss-Newton
Nonlinear Conjugate Gradient
Steepest descent
Misfit
Iteration
0
5
10
15
20
25
30
35
40
45
0
50
100
150
200

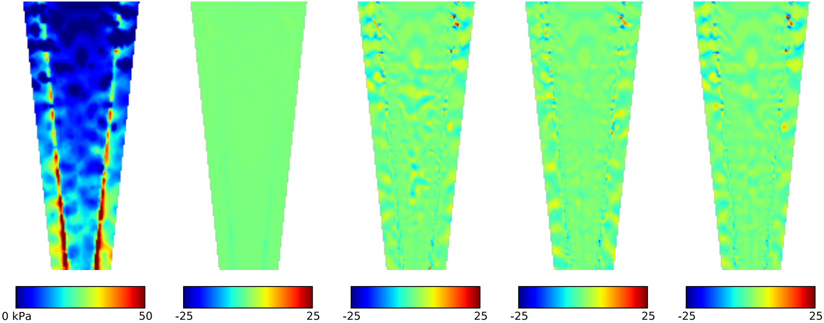
Reconstructions for 2d synthetic data

Reconstruction of basal stress starting with several initial estimates via steepest descent. (Joughin, MacAyeal, Tulaczyk '04)



Reconstructions for 2d synthetic data

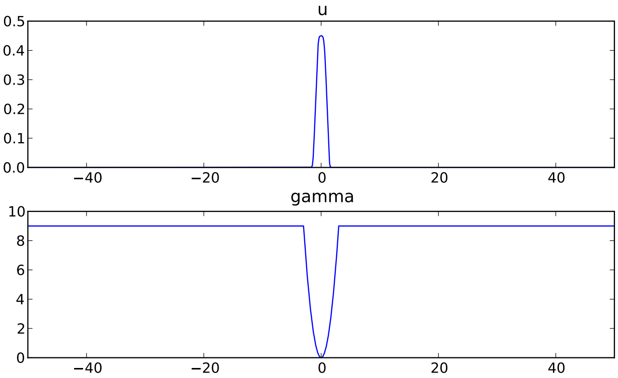
Reconstructions via incomplete Gauss-Newton.



What's not so good

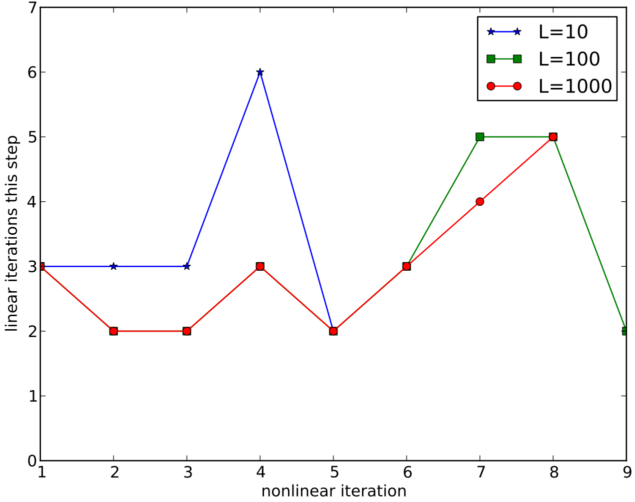
- For the applicability of adjoints, domain and range need to be Hilbert spaces. E.g. no L^1 or L^∞ norms allowed.
- You still need to fully solve a number of nonlinear forward problems.
- I don't know how to tell you how to pick δ.
- There's no proof that any of this works.

A lonely ice stream

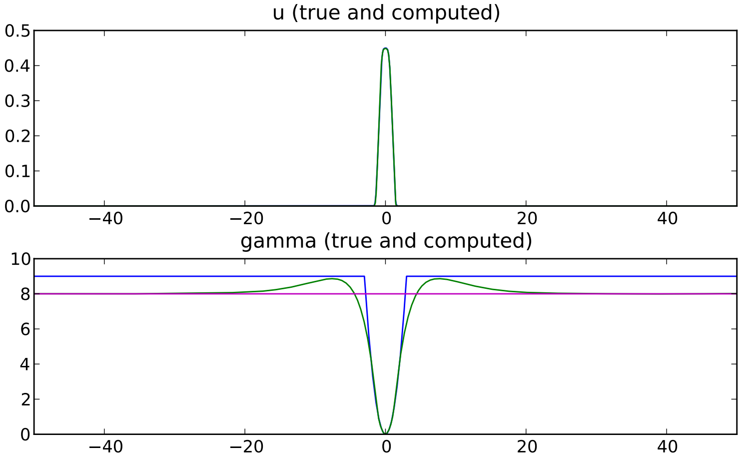


Ice stream is embedded in a larger sheet of length L varying from $L = 10$, to $L = 1000$.

Iteration counts



Lonely ice stream reconstructions



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