What is ... a \textit{variational inequality}?

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what is ... the source of my title?

WHAT IS... 

a Gröbner Basis?  
Bernd Sturmfels

WHAT IS... 

a Quasi-morphism?  
D. Kotschick

WHAT IS... 

a Random Matrix?  
Persi Diaconis

WHAT IS... 

a Systole?  
Marcel Berger
Outline

problems you can write as variational inequalities

obstacle problem example

three variational inequalities for glaciers
suppose you have a smooth function and you want to minimize it
• suppose you have a smooth function *on a closed, bounded interval*
• and you want to minimize it
because $f$ is smooth, you can say about the minimizer $u$ that:

- if $a < u < b$ then $f'(u) = 0$ or
- if $u = a$ then $f'(u) \geq 0$ or
- if $u = b$ then $f'(u) \leq 0$
because \( f \) is smooth, you can say about the minimizer \( u \) that: the \textcolor{red}{\textbf{variational inequality}} applies,

\[
f''(u)(v - u) \geq 0 \quad \text{for all } v \in [a, b]
\]
what is a variational inequality?

1. cute way to rewrite a calc problem in $\mathbb{R}^1$; min $f$ solves:

$$u : \quad f'(u)(v - u) \geq 0 \quad \text{for all } v \in [a, b]$$

the above is a necessary condition only
projection onto a closed, convex set $K \subset \mathcal{H}$

Suppose $K \subset \mathbb{R}^n$ is closed and convex. (Or $K \subset \mathcal{H}$ closed and convex, where $\mathcal{H}$ is a Hilbert space.)

Definition. Given $x \in \mathbb{R}^n$ (or $x \in \mathcal{H}$), the unique minimizer

$$y = \min_{z \in K} \|x - z\|$$

is the projection of $x$ onto $K$, written $y = \Pi x$. 
projection onto a closed, convex set $K \subset \mathcal{H}$

Theorem.

$$y = \Pi x \iff (y - x) \cdot (\eta - y) \geq 0 \text{ for all } \eta \in K$$

This is also a variational inequality.

Idea. The angle between $y - x$ and $\eta - y$ is $\leq 90^\circ$ for all $\eta \in K$. 
Theorem.

\[ y = \Pi x \iff (y - x) \cdot (\eta - y) \geq 0 \quad \text{for all} \quad \eta \in K \]

This is also a variational inequality.

Idea. The angle between \( y - x \) and \( \eta - y \) is \( \leq 90^\circ \) for all \( \eta \in K \).
what is a variational inequality?

1. cute way to rewrite a calc problem in $\mathbb{R}^1$; min $f$ solves:
   
   $$ u : \quad f'(u)(v - u) \geq 0 \quad \text{for all } v \in [a, b] $$

2. dot-product for projection on a closed, convex $K \subset \mathcal{H}$:
   
   $$ y = \Pi x : \quad (y - x) \cdot (\eta - y) \geq 0 \quad \text{for all } \eta \in K $$
Consider a mainstream math problem:

- let $K \subset \mathbb{R}^n$ be convex
- let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth ($C^1$)
- find $u \in K$ so that $f$ is minimum?
minimize $f$ on convex $K \subset \mathbb{R}^n$

Consider a mainstream math problem:

- let $K \subset \mathbb{R}^n$ be convex
- let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth ($C^1$)
- find $u \in K$ so that $f$ is minimum?

Claim: if $u \in K$ minimizes $f$ then $u$ solves

$$\nabla f(u) \cdot (v - u) \geq 0 \quad \text{for all } v \in K$$

which is a variational inequality
minimize $f$ on convex $K \subset \mathbb{R}^n$

*Theorem.* $K \subset \mathbb{R}^n$ convex. $f : \mathbb{R}^n \to \mathbb{R}$ smooth. If $u \in K$ minimizes $f$ then

$$\nabla f(u) \cdot (v - u) \geq 0 \quad \text{for all } v \in K.$$

*Proof.* If $0 \leq t \leq 1$ then

$$(1 - t)u + tv \in K$$

But $t = 0$ is minimizer of

$$g(t) = f ((1 - t)u + tv)$$

so by the “calculus I problem”, $g'(0) \geq 0$. By chain rule

$$g'(t) = (\nabla f)((1 - t)u + tv) \cdot (-u + v)$$

so $g'(0) = (\nabla f)(u) \cdot (v - u) \geq 0$. \qed
minimize $f$ on convex $K \subset \mathbb{R}^n$

How about existence of $u$?
It happens if either:
- $K$ is compact
- $K$ is closed and $f$ is coercive

How about uniqueness of $u$?
It happens if:
- $f$ is strictly convex
what is a variational inequality?

1. cute way to rewrite a calc problem in $\mathbb{R}^1$; min $f$ solves:
   $$u : \quad f'(u)(v - u) \geq 0 \quad \text{for all } v \in [a, b]$$

2. dot-product for projection on a closed, convex $K \subset H$:
   $$y = \Pi x : \quad (y - x) \cdot (\eta - y) \geq 0 \quad \text{for all } \eta \in K$$

3. rewriting of “min $f$ on convex $K \subset \mathbb{R}^n$”:
   $$u : \quad \nabla f(u) \cdot (v - u) \geq 0 \quad \text{for all } v \in K$$
solve \( F(x) = 0 \) on \( \mathbb{R}^n \)

Consider another mainstream applied math problem:

- Given continuous function \( F : \mathbb{R}^n \to \mathbb{R}^n \), find \( x \in \mathbb{R}^n \) so that
  \[ F(x) = 0. \]

- That is, solve \( n \) nonlinear equations in \( n \) unknowns.
solve $F(x) = 0$ on $\mathbb{R}^n$

Consider another mainstream applied math problem:

- Given continuous function $F : \mathbb{R}^n \to \mathbb{R}^n$, find $x \in \mathbb{R}^n$ so that $F(x) = 0$.
- That is, solve $n$ nonlinear equations in $n$ unknowns.
- *No one has anything positive to say about this problem:* no guarantee of existence, no guarantee of uniqueness, no effective theory of approximation.
solve $F(x) = 0$ on compact, convex $K \subset \mathbb{R}^n$

But we can change the problem minimally, and have something positive to say:

- Assume $K \subset \mathbb{R}^n$ is compact and convex.
- Seek $x \in K$ so that $F(x) = 0$.
- Theorem. There is $x \in K$ so that

$$F(x) \cdot (y - x) \geq 0 \quad \text{for all } y \in K$$

which is a variational inequality
solve \( F(x) = 0 \) on compact, convex \( K \subset \mathbb{R}^n \)

Theorem. There is \( x \in K \) so that

\[
F(x) \cdot (y - x) \geq 0 \quad \text{for all } x \in K.
\]

Proof. \( x' \mapsto \Pi(x' - F(x')) \), as map on \( K \), has a fixed point.
what is a *variational inequality*?

1. cute way to rewrite a calc problem in \( \mathbb{R}^1 \); \( \min f \) solves:
   
   \[
   u : \quad f'(u)(v - u) \geq 0 \quad \text{for all } v \in [a, b]
   \]

2. dot-product for projection on a closed, convex \( K \subset \mathcal{H} \):
   
   \[
   y = \Pi x : \quad (y - x) \cdot (\eta - y) \geq 0 \quad \text{for all } \eta \in K
   \]

3. rewriting of “\( \min f \) on convex \( K \subset \mathbb{R}^n \)”:
   
   \[
   u : \quad \nabla f(u) \cdot (v - u) \geq 0 \quad \text{for all } v \in K
   \]

4. gets existence for *any* continuous nonlinear eqns on compact, convex \( K \subset \mathbb{R}^n \):
   
   \[
   x : \quad F(x) \cdot (y - x) \geq 0 \quad \text{for all } y \in K
   \]
on convex $K$

if $K$ is convex and $u, v \in K$ and $0 \leq \varepsilon \leq 1$ then

$$(1 - \varepsilon)u + \varepsilon v = u + \varepsilon(v - u) \in K$$

$(1 - \varepsilon)u + \varepsilon v$ as a linear combination in $K$

$u + \varepsilon(v - u)$ as a vector from $u$ directed into $K$
Outline

problems you can write as variational inequalities

obstacle problem example

three variational inequalities for glaciers
elastic membrane over obstacle

- elastic membrane $z = u(x, y)$ minimizes energy

$$J[v] = \int_{\Omega} \frac{1}{2} |\nabla v|^2 - f v$$

where $f$ is upward force on the membrane

- if surface $v(x, y)$ is above an obstacle $\psi(x, y)$ then it’s in convex set

$$\mathcal{K} = \{ v \in H^1_0(\Omega) : v \geq \psi \}$$
variational inequality for obstacle problem

- if \( u \in \mathcal{K} \) is minimizer and if \( v \in \mathcal{K} \) and if \( 0 \leq \epsilon \leq 1 \) then

\[
0 \leq J[u + \epsilon(v - u)] - J[u] = \epsilon \int_{\Omega} \nabla u \cdot \nabla (v - u) - f(v - u) + \epsilon^2 \int_{\Omega} |\nabla (v - u)|^2
\]

- thus as \( \epsilon \to 0 \), we know that \( u \in \mathcal{K} \) satisfies

\[
\int_{\Omega} \nabla u \cdot \nabla (v - u) - f(v - u) \geq 0 \quad \forall v \in \mathcal{K}
\]

which is the variational inequality formulation

- also written:

\[
\langle \nabla J(u), v - u \rangle \geq 0 \quad \forall v \in \mathcal{K}
\]
PDE for obstacle problem

- where $u > \psi$, the variational inequality implies $-\nabla^2 u = f$
  - the standard PDE for an *unobstructed* elastic membrane
  - $-\nabla^2 u = f$ is “Poisson equation”

- an engineer would say

  \[ \text{the membrane } u(x, y) \text{ solves } -\nabla^2 u = f \text{ except when it is in contact with the obstacle} \]

- but the set on which the contact happens is *a priori* unknown . . .
finite elements

- the finite element method (FEM) was built on variational \textit{equalities}, i.e. “weak formulations”
- so variational inequalities play well with FEM
- FEM represents (approximates) function spaces on pretty meshes like this:
the 3-point, one-dimensional obstacle problem

- for example, consider a one-dimensional obstacle problem
- with an equally-spaced 3-point mesh
- and a constant force \( f(x) = f_0 \)
- so \( \Omega = [0, L] \) has mesh points \( \{x_1, x_2, x_3\} \)
- the energy is just a quadratic function in \( \mathbb{R}^3 \):

\[
J[v] = \int_0^L \frac{1}{2} (v')^2 - f_0 v \\
\approx \frac{1}{\Delta x} \left( v_1^2 + v_2^2 + v_3^2 - v_1 v_2 - v_2 v_3 \right) - f_0 \Delta x (v_1 + v_2 + v_3)
\]

- and \( \psi(x) \) and \( u(x) \) represented by just three values each
a 3-point case of the obstacle problem

1: zero force $f_0 = 0$, one-hump obstacle
a 3-point case of the obstacle problem

2: upward force $f_0 > 0$, flat obstacle
a 3-point case of the obstacle problem

3: downward force $f_0 < 0$, two-peak obstacle
a 3-point case of obstacle problem

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three variational inequalities for glaciers
the steady-climate question for ice sheets

- suppose a steady-state climate
- where it snows some places and melts in others
- the ice flows into areas where there is melting
- questions:
  - what land is covered by ice sheets?
  - how thick are these sheets?
ice sheet geometry: an obstacle analogy

- **ice surface** $s(x, y)$
  $\sim$ membrane
- **bedrock** $b(x, y)$
  $\sim$ obstacle
v.i. 1: steady ice sheet surface “equation”

- ice sheet surface equation (so-called “SIA”) applies only on domain where \( s > b \)
- let \( h = s - b \), the ice sheet thickness
- equation applies only where \( s > b \iff h > 0 \)
- define \( p = n + 1 \) where \( n \approx 3 \) for shear-thinning ice
- change variables \( h = u^{(p-1)/(2p)} \)
- define convex set \( K = \{ v \in W^{1,p}_0(\Omega), v \geq 0 \} \)
- Theorem (Jouvet-Bueller 2012). There is \( u \in K \) solving the steady transformed SIA,

\[
\int_{\Omega} (\mu|\nabla u - \Phi(u)|^{p-2}(\nabla u - \Phi(u))) \cdot \nabla (v - u) \geq \int_{\Omega} \alpha(u)(v - u)
\]

for all \( v \in K \)
marine ice sheets: overview

- marine ice sheets are full of free boundaries:
  - boundary between floating ("shelf") and grounded
  - boundary between sliding ("stream") and not ("sheet")
  - boundary between wet base and dry base
- the Antarctic ice sheet is a marine ice sheet

[Cartoon from (Schoof, 2006)]
[Cartoon from (Martin et al., 2011)]
ice stream sliding: an analogy

- ice stream is a viscous membrane with basal stresses
- ice streams emerge where basal resistance is sufficiently low
- a basal resistance model:
  - Coulomb friction, with
  - a yield stress distribution $\tau_c$
- Schoof’s slider analogy
v.i. 2: ice stream velocity “equation”

- let \( q = 1 + \frac{1}{n} \) where \( n \approx 3 \) for shear-thinning ice
- \( V \) is ice stream velocity, \( f = -\rho gh \nabla s \) is driving stress, \( F \) is lateral stress along calving front
- Theorem (C. Schoof, 2006). There is unique velocity \( U = (u, v) \in W^{1,q}(\Omega) \) solving the coulomb ice stream problem. It minimizes

\[
J[V] = \int_{\Omega} \frac{2B}{q} h \|V\|^q + \tau_c |V| - f \cdot V - \int_{\partial\Omega} F \cdot V
\]

with no constraint
- but \( J[V] \) is not smooth because of “\( \tau_c |V| \)”
v.i. 2: ice stream velocity “equation”

- Schoof started with a PDE for ice stream velocity (MacAyeal, 1989)
- then derived the variational inequality form: \( U \in W^{1,q}(\Omega) \) solves

\[
\int_{\Omega} T_{ij}(U) D_{ij} (V - U) + \tau_c (|V| - |U|) - f \cdot (V - U) \geq \int_{\partial \Omega} F \cdot (V - U)
\]

for all \( V \in W^{1,q}(\Omega) \)

- and then got \( J[V] \) on previous slide
some glaciers have cold ice

- to a glaciologist, ice is “cold” or “temperate”
  - cold ice has temperature below \(0^\circ C\)
  - temperate ice is at \(0^\circ C\), but with liquid water

- temperature \(u\), flow velocity \(V\)

- heat flux is \(q = -k \nabla u + \rho c V u\)
  - conductive flux nearly zero in temperate ice (\(\nabla u \approx 0\))

- viscous dissipation causes heating at rate \(S\)
v.i. 3: cold-ice temperature in polythermal glacier

- in cold ice, everyone knows the steady temperature solves

\[ 0 = \nabla \cdot (k \nabla u - \rho c V u) + S \]

- but where is the free boundary of the cold ice? (the “CTS”)

- define convex set

\[ \mathcal{K} = \{ \phi \in H^1(\Omega) \mid \phi \leq 0, \phi \text{ satisfies b.c.s} \} \]

- Theorem (Gillispie-Bueler, in prep). there exists \( u \in \mathcal{K} \) s.t.

\[ \int_{\Omega} (k \nabla u - \rho c V) \cdot \nabla (\phi - u) \geq \int_{\Omega} S(\phi - u) \]

for all \( \phi \in \mathcal{K} \)
variational inequalities for ice: a summary

- of the three variational inequalities:
  - 1 for the ice sheet surface is *not* a minimization
  - 2 for ice stream sliding is an unconstrained minimization, but of a non-smooth functional
  - 3 for the cold ice is *not* a minimization

variational inequalities will be used in future glacier and ice sheet problems because of all the free boundaries when a glaciologist says "this equation describes this..." they mean "this equation describes this... wherever the equation can be applied and I can't generally tell you where that is"

this makes the job of building an ice sheet model harder

and that's it on variational inequalities for today
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variational inequalities for ice: a summary

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• variational inequalities will be used in future glacier and ice sheet problems because of all the free boundaries between different equations
  ◦ when a glaciologist says “this equation describes this . . .” they mean “this equation describes this . . . wherever the equation can be applied and I can’t generally tell you where that is”
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