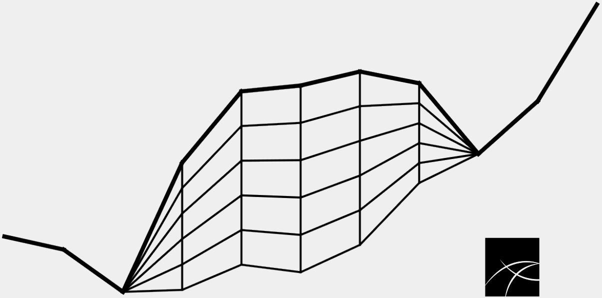
Evolution of ice sheet geometry using Stokes dynamics

Ed Bueler



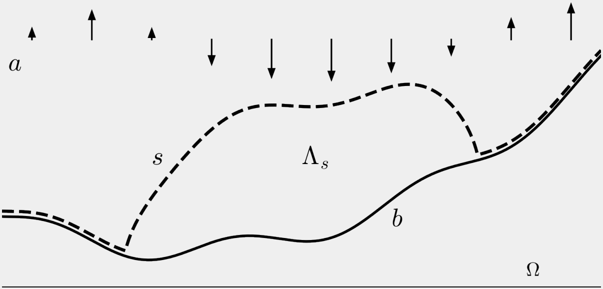
SIAM GS21

UNIVERSITY of ALASKA
Many Traditions One Alaska

1. explain the problem I’m working on
2. mention the (current) barriers to success

- work in progress
- I am giving this talk at 4am my time
  - low expectations, please

THE ICE GEOMETRY PROBLEM (IGP)



- how does glacier geometry evolve in response to the climate?
- only the simplest version (grounded, nonsliding, isothermal)
- assume time-independent data on fixed $\Omega \subset \mathbb{R}^2$:
  - (net) climatic mass balance $a(x, y)$ $\leftarrow$ *$a > 0$ for accumulation*
  - bed elevation $b(x, y)$
- *to compute*: surface elevation $s(t, x, y)$ on Ω
  icy domain $\Lambda_s \subset \mathbb{R}^3$

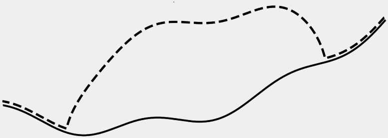
- glaciers flow, so one must solve the surface kinematical equation (SKE) *on* the ice

$$\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a = 0$$

  - $\mathbf{n}_s = \langle -s_x, -s_y, 1 \rangle$ is an upward normal to the ice surface
  - where *not* on the ice: $a \leq 0$

- assumed:
  - s well-defined (no overhangs)
  - $s = b$ where ice is not present

    $\therefore$ s is defined on all of Ω



- the steady IGP is a nonlinear complementarity problem (NCP) for s, a free-boundary problem, which is coupled to a non-Newtonian Stokes problem for $\mathbf{u}$, p:

$$\left.\begin{aligned} s - b &\geq 0 && \text{on } \Omega \\ -\mathbf{u}|_s \cdot \mathbf{n}_s - a &\geq 0 && \text{''} \\ (s-b)(-\mathbf{u}|_s \cdot \mathbf{n}_s - a) &= 0 && \text{''} \end{aligned}\right\} \text{NCP}$$

$$\left.\begin{aligned} -\nabla \cdot (2\nu_\epsilon D\mathbf{u}) + \nabla p - \rho_i \mathbf{g} &= \mathbf{0} && \text{on } \Lambda_s \\ \nabla \cdot \mathbf{u} &= 0 && \text{''} \\ \mathbf{u} &= \mathbf{0} && \text{on } \Gamma_0 \quad \text{(ice base)} \\ (2\nu_\epsilon D\mathbf{u} - pI)\,\mathbf{n} &= \mathbf{0} && \text{on } \partial\Lambda_s \setminus \Gamma_0 \end{aligned}\right\} \text{Stokes}$$

  - Glen-law effective viscosity with $\mathrm{p} = (1/\mathrm{n}) + 1 (= 4/3)$:

$$\nu_\epsilon = \frac{1}{2} B_n \left(|D\mathbf{u}|^2 + \epsilon\, D_0^2 \right)^{(\mathrm{p}-2)/2}$$

  - *the 3 nonlinearities*

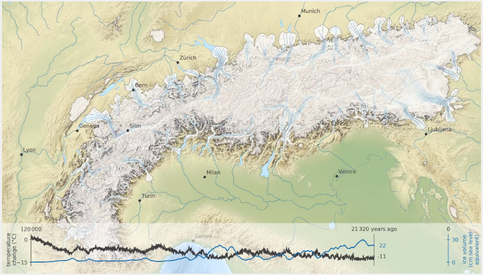
IMPLICIT IGP STRONG FORM

- solve one step of backward Euler for $s, \mathbf{u}, p$ at new time t^ℓ
  - $\frac{\partial s}{\partial t} \approx \frac{s^\ell - s^{\ell-1}}{\Delta t}$ in evolving SKE, and write s for s^ℓ
- at each time step, solve NCP coupled to Stokes:

$$
\begin{aligned}
s - b &\geq 0 && \text{on } \Omega \\
s - s^{\ell-1} - \Delta t\, \mathbf{u}|_s \cdot \mathbf{n}_s - \Delta t\, a &\geq 0 && \text{"} \\
(s - b)(s - s^{\ell-1} - \Delta t\, \mathbf{u}|_s \cdot \mathbf{n}_s - \Delta t\, a) &= 0 && \text{"} \\
-\nabla \cdot (2\nu_\epsilon D\mathbf{u}) + \nabla p - \rho_\text{i}\mathbf{g} &= \mathbf{0} && \text{on } \Lambda_s \\
\nabla \cdot \mathbf{u} &= 0 && \text{"} \\
\mathbf{u} &= \mathbf{0} && \text{on } \Gamma_0 \\
(2\nu_\epsilon D\mathbf{u} - pI)\,\mathbf{n} &= \mathbf{0} && \text{on } \partial\Lambda_s \setminus \Gamma_0
\end{aligned}
$$

  - very similar to steady IGP

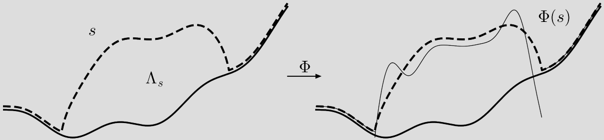
- almost no one is solving such a implicit or steady IGP
  - except Bueler (2016), Brinkerhoff et al (2017) (*shallow*)
  - and Wirbel & Jarosch (2020) (*semi-coupled, fake ice layer*)
  - none are scalable (*single level*)
- what are people doing instead, for production science?
  - explicit time-stepping with $s \geq b$ enforced by truncation
  - usually shallow approximations, e.g. PISM run below



by Julien Seguinot

ice dynamics operator

$\Phi : s \mapsto -\mathbf{u}|_s \cdot \mathbf{n}_s$



- to compute $\Phi(s)$: build Λ_s, solve weak-form Stokes problem

$$F_{\Lambda_s}(\mathbf{u}, p)[\mathbf{v}, q] = \int_{\Lambda_s} 2\nu_\epsilon D\mathbf{u} : D\mathbf{v} - p\nabla\cdot\mathbf{v} - (\nabla\cdot\mathbf{u})q - \rho_i \mathbf{g}\cdot\mathbf{v}\, d\mathbf{x} = 0,$$

  extract trace $\mathbf{u}|_s$, then extend $\Phi(s) = -\mathbf{u}|_s \cdot \mathbf{n}_s$ by zero to Ω
- regarding this Stokes problem (for *fixed* geometry):
  - well-posed over $W_0^{1,p}(\Lambda_s)^3 \times L^q(\Lambda_s)$ (Jouvet & Rappaz 2011)
  - optimal solver exists (Isaac et al 2015)

$\therefore$ $\phi(s)$ is well-defined if s is piecewise C^1
  - which is a regularity conjecture

IGP WEAK FORM IS A VARIATIONAL INEQUALITY

- using Φ gives a cleaner NCP over Ω for the steady IGP:

$$\begin{aligned} s - b &\geq 0 \\ \Phi(s) - a &\geq 0 \\ (s - b)(\Phi(s) - a) &= 0 \end{aligned}$$

  - recall NCP (strong form) $\leftrightarrow$ VI (weak form)

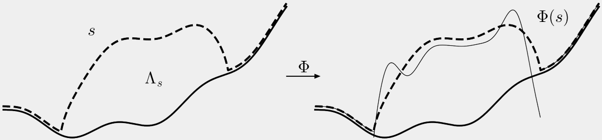
steady IGP weak form

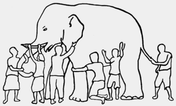
find admissible $s \in \mathcal{K} = \{r \geq b\} \subset W^{1,q}(\Omega)$ so that

$$(\Phi(s) - a, r - s) \geq 0 \qquad \text{for all } r \in \mathcal{K}$$

- well-posedness is almost completely open
  - existence holds for SIA version (Jouvet & Bueler, 2012)

WHAT KIND OF BEAST IS Φ?





- $\Phi : \mathcal{K} \subset W^{1,q}(\Omega) \to W^{-1,p}(\Omega)$ where $q = 4$ and $p = 4/3$
- the Stokes stress balance is nonlocal
  - Φ is integral operator (*convolve with Stokeslets*)
  - Φ is short range (*dense but concentrated near the diagonal*)
- $\Phi \approx \Phi_{SIA}$ for shallow glaciers, a differential operator:

$$\Phi_{SIA}(s) = -\frac{\gamma}{q}(s-b)^q|\nabla_2 s|^q - \nabla_2 \cdot \left[\frac{\gamma}{q+1}(s-b)^{q+1}|\nabla_2 s|^{q-2}\nabla_2 s\right]$$

  - Φ is elliptic “in the large”
  - Φ is degenerate at the ice margin
- yet common to say
  - Φ is advective because SKE is write-able as a transport equation for thickness: $H_t + \bar{\mathbf{u}}\,\nabla H = a$

GOAL: ROBUST AND OPTIMAL IGP SOLVER

numerical problem

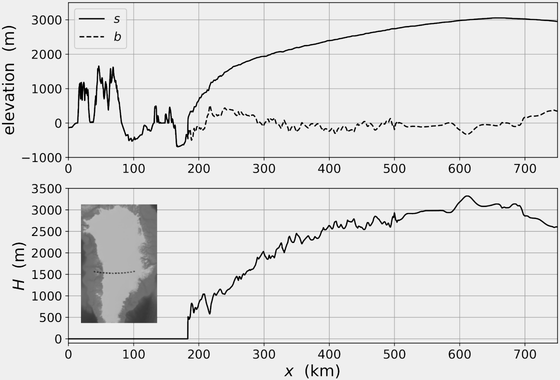
given triangulation $\mathcal{T}$ of Ω and P_1 space $\mathcal{V}^h \subset W^{1,q}(\Omega)$, and convex set $\mathcal{K}^h = \{r^h \geq b^h\} \subset \mathcal{V}^h$, solve VI for $s^h \in \mathcal{K}^h$:

$$\left(\Phi^h(s^h) - a, r^h - s^h\right) \geq 0 \qquad \text{for all } r^h \in \mathcal{K}^h$$

- P_1 because of low regularity at free boundary
- evaluate $\Phi^h(s^h) = -\mathbf{u}|_{s^h} \cdot \mathbf{n}_{s^h}$ by solving Stokes problem $F_{\Lambda_{s^h}}(\mathbf{u}^h, p^h) = 0$

- seeking a solver which is:
  - robust
  - near optimal $O(m^{1+\epsilon})$ work over m degrees of freedom in $\mathcal{V}^h$
    - requires multigrid . . . more below
  - *these are aspirations*

- which solution variable?
  - surface elevation s?
  - thickness H?



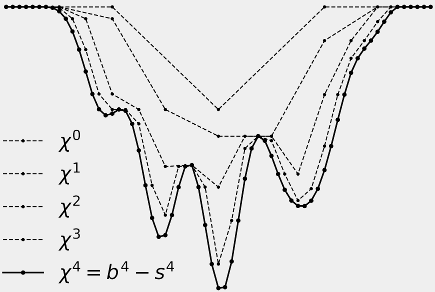
- smoothness suggests s is better for multigrid
  - but *do not* put “fake ice” where $s = b$; only solve for ice velocity where there is ice!

MULTILEVEL CONSTRAINT DECOMPOSITION (MCD)

strategy

decompose $\mathcal{K}^h$ using mesh hierarchy and then apply V-cycles

- mesh levels over Ω:
  $\mathcal{T}^0$ (coarse) to $\mathcal{T}^J$
- given fine-level iterate s^J
- *defect constraint* $\chi^J = b^J - s^J$
- *monotone restriction* $R^\oplus$
- defect constraints on levels:
  $\chi^j = R^\oplus \chi^{j+1}$
- admissible perturbations:
  $z^j \geq \chi^j \iff s^j + z^j \geq b^j$



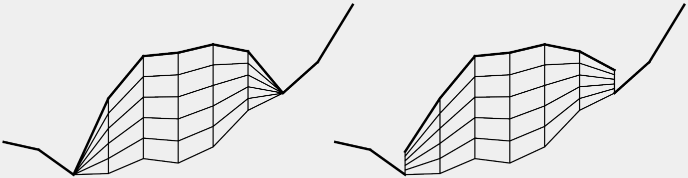
MULTILEVEL CONSTRAINT DECOMPOSITION (MCD)

- in V-cycle, on each level solve a VI:

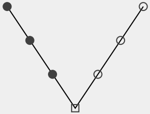
$$\left(\Phi^j(s^j + z^j) - a^j, v^j - z^j\right) \geq 0 \quad \text{for all } v^j \geq \chi^j$$

- this strategy gives an $O(m \log m)$ solver for the Laplacian obstacle problem (Tai, 2003)
  - a multilevel VI strategy is the only hope for (near) optimality
- for IGP:
  - for nonlinear VI, need additional FAS strategy ✓
  - . . . but each residual is nonlocal and expensive

STOKES ON A FIREDRAKE EXTRUDED MESH



- V-cycle based on mesh hierarchy $\{\mathcal{T}^j\}$ over Ω
- to evaluate the residual $\Phi(s^j) - a^j$ in the VI:
  - extrude base mesh $\mathcal{T}^j$ to current geometry s^j
    - with or without small cliffs
  - no extrusion where ice free
    - thanks to Lawrence Mitchell for this capability
  - solve Stokes problem $F_{\Lambda_{s^j}}(\mathbf{u}^j, p^j) = 0$
  - compute $\Phi(s^j) = -\mathbf{u}^j|_{s^j} \cdot \mathbf{n}_{s^j}$ and subtract a^j



SMOOTHER OPTIONS

- now all the evil is in the smoother
- smoother options from weaker to stronger:
  - nonlinear Richardson?
  - exploit SIA analogy?
    - pointwise smoothers work for SIA
  - nonlinear pointwise smoothers (GS, Jacobi)?
    - GS extremely expensive because residual nonlocal
    - seem to fail
  - reduced-space Newton using banded Jacobian approximation?
    - finite-difference Jacobian entries using pseudo-coloring

- nothing works well so far

CONCLUSION

- *GOAL 1*: no shallow approximations in flow physics
  - ✓ Stokes solver working in Firedrake on extruded mesh
  - ✓ solver choices by (Isaac et al, 2015) putatively optimal
- *GOAL 2*: beat explicit time-stepping
  - ✓ observe that problem is
    - constrained (nonlinear VI on $\mathcal{K} = \{s \geq b\}$)
    - diffusive in the large (Stokes $\approx$ SIA)
    - nonlocal (Stokes)
  - ✓ MCD FAS path to optimality for solving the VI
  - :-(stuck on finding a robust, efficient smoother

ultimate goal

for better science, enable long-time (paleo) simulations without shallow approximations

thanks for your attention . . . questions?

Some references 1

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NCP is a barrier to exact numerical mass conservation[1]

[1] E. Bueler. **Conservation laws for free-boundary fluid layers.** *SIAM J. Appl. Math., to appear.*