Implicit multigrid methods for glacier geometry

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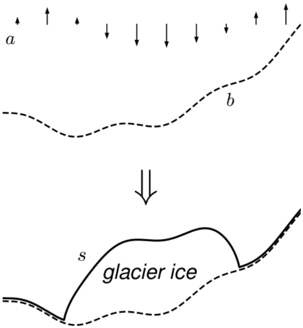
SIAM AN22



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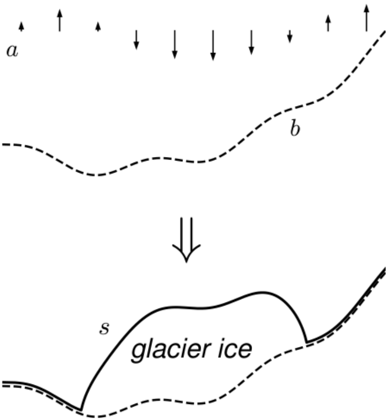
OUTLINE

1. model for evolving glacier surfaces
2. implicit-step free-boundary problem (NCP or VI)
3. computational performance model
4. geometric multigrid with distinctive features
    - constraint decomposition
    - FAS-type nonlinear coarse corrections
5. minimal results, smoother difficulties

- inputs:
  - a = climatic mass balance (snowfall or melt&runoff)
  - b = bed elevation
  - other inputs ignored for simplicity: surface temperature, bed composition, geothermal, . . .
- goal: construct fast numerical methods for this map!

$$\begin{pmatrix}\text{climate and}\\ \text{topography}\end{pmatrix} \to \begin{pmatrix}\text{glacier}\\ \text{geometry}\end{pmatrix}$$

$$a, b \quad \mapsto \quad s$$



- glacier geometry would be easy if the snow just piled-up!
- but glaciers flow
  - very viscous, non-Newtonian fluid driven by gravity
  - flow is mostly downhill, along $-\nabla_{\mathbf{x}} s$
  - ice flows into areas where there is melt

- $\mathbf{x} = (x, y)$ denotes horizontal coordinates
  - $\mathbf{x}$ is map-plane
- data given on 2D domain Ω:
  - climate (time-dependent): $a(t, \mathbf{x})$
  - bed elevation (stationary): $b(\mathbf{x})$
- we seek surface elevation $s(t, \mathbf{x})$
  - obviously glaciers are on top: $s(t, \mathbf{x}) \geq b(\mathbf{x})$
- also seek velocity $\mathbf{u}(t, \mathbf{x}, z)$ within 3D ice
  - $\Lambda_s := \{b(\mathbf{x}) < z < s(t, \mathbf{x})\}$

- mass conservation at glacier surface?
- this is the surface kinematical equation:

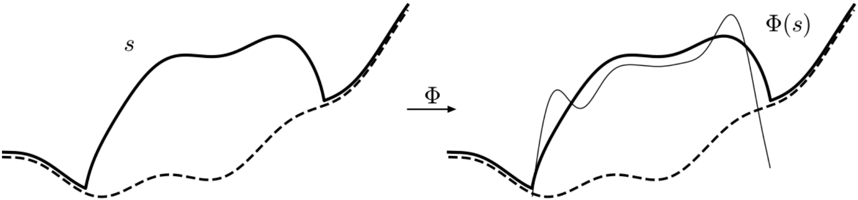
$$\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a = 0$$

  - $\mathbf{n}_s = \langle -\nabla_{\mathbf{x}} s, 1 \rangle$ is upward normal to surface
  - $\mathbf{u}|_s = \mathbf{u}(t, \mathbf{x}, s(t, \mathbf{x}))$ is 3D velocity at surface

- we will regard flow as a Stokes problem
  - or a shallow approximation thereof
- problem becomes clearer if we abstract the flow as a map:

$$\begin{pmatrix}\text{surface}\\ \text{elevation}\end{pmatrix} \to \begin{pmatrix}\text{ice motion}\\ \text{from dynamics}\\ \text{at surface}\end{pmatrix}$$

$$\Phi : \quad s \mapsto -\mathbf{u}|_s \cdot \mathbf{n}_s$$



HOW TO COMPUTE $\Phi(s)$?

- how to compute $\Phi(s) = -\mathbf{u}|_s \cdot \mathbf{n}_s$?

- Stokes: solve for $\mathbf{u}$:

$$\int_{\Lambda_s} 2\nu_\epsilon(D\mathbf{u})D\mathbf{u} : D\mathbf{v} - p\nabla\cdot\mathbf{v} - (\nabla\cdot\mathbf{u})q - \rho_\text{i}\mathbf{g}\cdot\mathbf{v}\,d\mathbf{x} = 0,$$

  over all test $\mathbf{v}$, q, where $\Lambda_s = \{(\mathbf{x}, z) : b(\mathbf{x}) < z < s(t, \mathbf{x})\}$ is the 3D ice and

$$\nu_\epsilon(D\mathbf{u}) = \frac{1}{2}\Gamma\left(|D\mathbf{u}|^2 + \epsilon\, D_0^2\right)^{(\text{p}-2)/2},$$

  is effective viscosity and $\text{p} = \frac{1}{\text{n}} + 1$, then extract trace $\mathbf{u}|_s$, then extend $\Phi(s) = -\mathbf{u}|_s \cdot \mathbf{n}_s$ by 0 to Ω
  - well-posed over $W_0^{1,\text{p}}(\Lambda_s)^3 \times L^\text{q}(\Lambda_s)$ (Jouvet & Rappaz 2011)
  - near-optimal solver (Isaac et al 2015)

HOW TO COMPUTE $\Phi(s)$?

- how to compute $\Phi(s) = -\mathbf{u}|_s \cdot \mathbf{n}_s$?

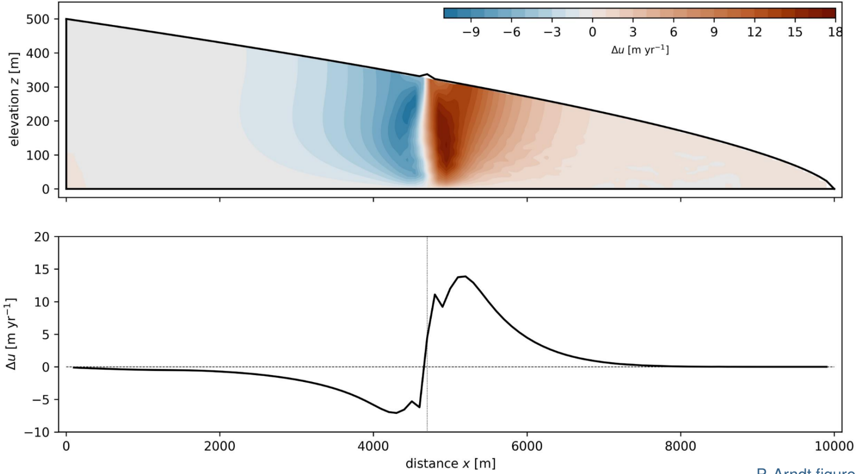
- shallow ice approximation (SIA): evaluate

$$\Phi(s) = -\frac{\gamma}{q}(s-b)^q|\nabla_{\mathbf{x}} s|^q - \nabla_{\mathbf{x}} \cdot \left(\frac{\gamma}{q+1}(s-b)^{q+1}|\nabla_{\mathbf{x}} s|^{q-2}\nabla_{\mathbf{x}} s\right)$$

  where $q = n + 1$ and $\gamma > 0$ is related to ice softness and density
  - SIA = lubrication approximation
  - $\Phi(s)$ is a nonlinear differential operator only because membrane (longitudinal) stresses are unbalanced

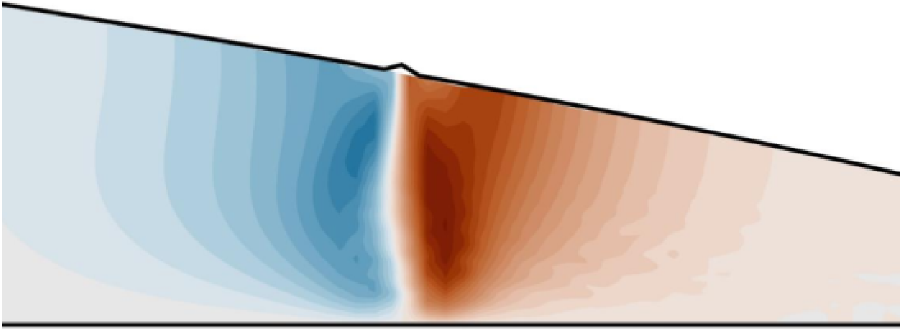
VELOCITY EXPRESSION OF s PERTURBATION

- derivative of $\Phi(s) = -\mathbf{u}|_s \cdot \mathbf{n}_s$?
- difference $\mathbf{u}_{(s+\psi)} - \mathbf{u}_{(s)}$ from surface perturbation ψ

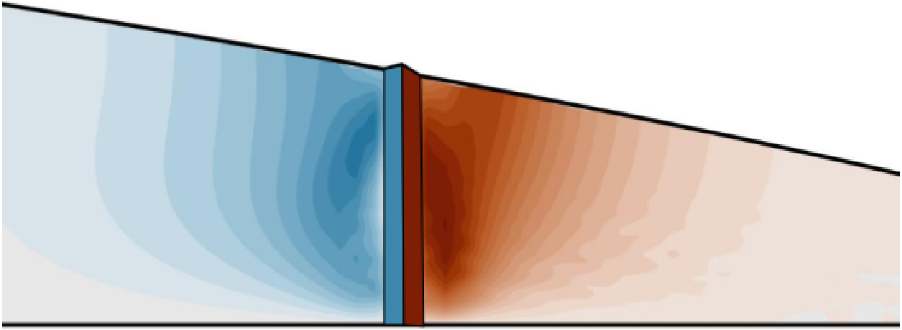


P. Arndt figure

- perturb *s* by 5 m bump, over 200 m
- resulting Stokes velocity difference
    - Stokes $\Phi(s)$ is not local
    - note longitudinal-stress range (Kamb & Echelmeyer 1986)



- perturb s by 5 m bump, over 200 m
- resulting Stokes velocity difference
  - Stokes $\Phi(s)$ is not local
  - note longitudinal-stress range (Kamb & Echelmeyer 1986)
  - overlain SIA difference, localized under s perturbation!



- goal: determine glacier surface elevation s and glaciated area
- this is a free-boundary problem in the map-plane
- basic logic gives a nonlinear complementarity* problem (NCP) over all of Ω:

$$\begin{aligned} s - b &\ge 0 \\ \frac{\partial s}{\partial t} + \Phi(s) - a &\ge 0 \\ (s-b)\left(\frac{\partial s}{\partial t} + \Phi(s) - a\right) &\overset{*}{=} 0 \end{aligned}$$

  - we extend $\mathbf{u}|_s$ by zero so $\Phi(s) = 0$ in ice-free areas
  - glaciated area $\Omega_+(t) := \{\mathbf{x} \,:\, s(t,\mathbf{x}) > b(\mathbf{x})\}$ . . . from solution!

BACKWARD EULER STEP

- ice mostly flows downhill, so SKE is mostly diffusive!
  - straightforward in shallow ice approximation
  - not literally a diffusion with Stokes dynamics
- implicit stepping makes sense ... try backward Euler:

$$\frac{s - s_o}{\Delta t} + \Phi(s) - a = 0$$

  - s_o is the previous surface elevation
  - s is new surface elevation
- each time step is a free boundary problem, an NCP:

$$\begin{aligned} s - b &\geq 0 \\ s + \Delta t\, \Phi(s) - (s_o + \Delta t\, a) &\geq 0 \\ (s - b)\Big(s + \Delta t\, \Phi(s) - (s_o + \Delta t\, a)\Big) &= 0 \end{aligned}$$

IMPLICIT STEP AS VARIATIONAL INEQUALITY

- convert to weak form for FE treatment: NCP $\Longrightarrow$ VI
- define admissible surface elevations:

$$\mathcal{K} = \{r \,:\, r \ge b\}$$

  - the constraint set
  - closed and convex subset of $\mathcal{V} = W^{1,p}(\Omega)$
- define nonlinear form and source linear functional:

$$N(s)[q] = \int_\Omega (s + \Delta t\, \Phi(s))\, q\, d\mathbf{x}, \quad f[q] = \int_\Omega (s_o + \Delta t\, a)\, q\, d\mathbf{x}$$

- backward Euler step VI: find $s \in \mathcal{K}$ so that

$$N(s)[r - s] \ge f[r - s]$$

for all $r \in \mathcal{K}$

DISCRETIZATION AND SOLVERS?

- VI $=$ abstract, dynamics-agnostic, implicit time-step problem:

$$N(s)[r - s] \geq f[r - s]$$

- next steps: FE discretization, solvers
- can we actually solve it?
  A. yes for the SIA (Bueler 2016)
  A. efficiently, some day, I hope, for Stokes
- what do current models do?
  A. explicit time stepping, usually forward Euler, and that is a tragedy

PISM PARALLEL ICE SHEET MODEL ISSM elmer ICE ICE PACK

- for this story I need a performance model (Bueler, submitted)
- use only 3 parameters to describe simulations:

	typical values
climate-couplings per model year q	0.1 – 12
domain size (length) L	1000 km – 10 km
mesh spacing Δx	10 km – 10 m

- stability restrictions for explicit schemes:
  - optimistic (CFL): $\Delta t < O\left(\frac{\Delta x}{U}\right)$ — weeks to hours?
  - pessimistic: $\Delta t < O\left(\frac{\Delta x^2}{D}\right)$ — days to minutes?
- degrees of freedom m scale with 2D mesh spacing:

$$\Delta x = O\left(\frac{L}{\sqrt{m}}\right) \quad \Longleftrightarrow \quad m = O\left(\frac{L^2}{\Delta x^2}\right)$$

ALGORITHMIC SCALING

- algorithmic scaling of solvers:
  - fixed-geometry velocity solution: $O(m^{1+\alpha})$ flops

    $\alpha \approx 0$ (multigrid: Brown, Isaac, Tuminaro), 1 (sparse direct)
  - implicit (velocity&geometry) solution: $O(m^{1+\gamma})$ flops

    $\gamma \approx$ unknown (Stokes), 0.8 (SIA: Bueler 2016)

- giving a model for asymptotic simulation cost:

time-stepping	*flops per model year*
explicit (optimistic)	$O\left(\frac{UL^{2+2\alpha}}{\Delta x^{3+2\alpha}}\right) = O\left(\frac{Um^{1.5+\alpha}}{L}\right)$
explicit (pessimistic)	$O\left(\frac{D\,L^{2+2\alpha}}{\Delta x^{4+2\alpha}}\right) = O\left(\frac{D\,m^{2+\alpha}}{L^2}\right)$
implicit	$O\left(\frac{q\,L^{2+2\gamma}}{\Delta x^{2+2\gamma}}\right) = O\left(q\,m^{1+\gamma}\right)$

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  - fixed-geometry velocity solution: $O(m^{1+\alpha})$ flops

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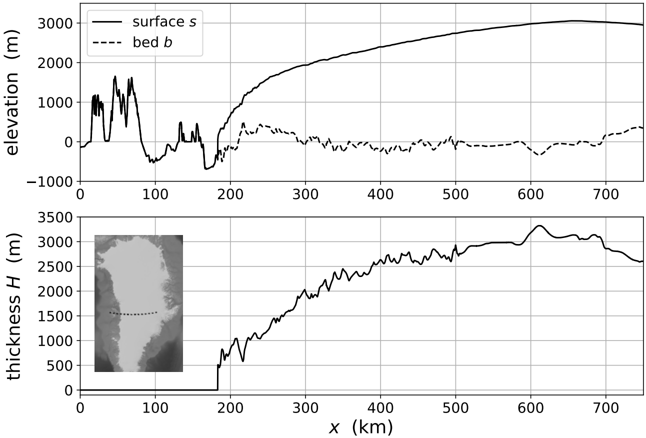
explicit (pessimistic)	$O\left(\dfrac{D\,L^{2+2\alpha}}{\Delta x^{4+2\alpha}}\right) = O\left(\dfrac{D\,m^{2+\alpha}}{L^2}\right)$	$\leftarrow$ tragedy
implicit	$O\left(\dfrac{q\,L^{2+2\gamma}}{\Delta x^{2+2\gamma}}\right) = O\left(q\,m^{1+\gamma}\right)$	

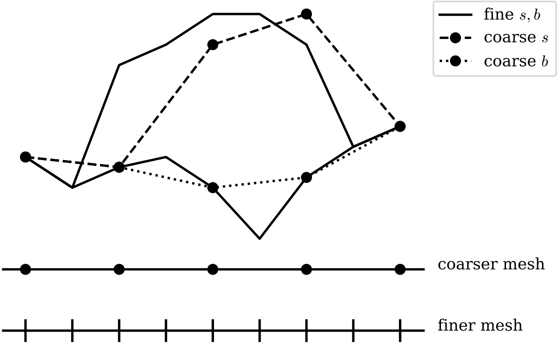
GEOMETRIC MULTIGRID FOR IMPLICIT STEPS?

- goal: $\gamma \approx 0$ in $O(m^{1+\gamma})$ cost of implicit solve
- can GMG be used as a solver?
  - need exploitable solution property: surface smoothness



GEOMETRIC MULTIGRID FOR IMPLICIT STEPS?

- solving a free-boundary (obstacle) problem
  - constraint $s \geq b$ (obstacle is rough bed elevation data!)
- issues:
  1. fine mesh bed data not present on coarse mesh
     - coarse iterates not necessarily admissible on fine
  2. free boundary location depends on mesh



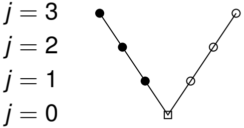
- next: outline a GMG for the glacier geometry problem
  - simultaneous solution for surface elevation and velocity

- distinctive flavors:
  1. subspace decomposition viewpoint
  2. ordinary residual does not converge to zero
  3. each coarse level has 2 constraint sets
     - defect constraint
     - monotone restriction
     - multilevel constraint decomposition (Tai, 2003)
  4. FAS-type nonlinear coarse corrections
  5. smoother?
     - Stokes residual is expensive and non-local!

SUBSPACE DECOMPOSITION VIEWPOINT

- hat functions span FE space on level j:

$$\mathcal{V}^j = \text{span}\{\psi_1^j(x), \ldots, \psi_{m_j}^j(x)\}$$



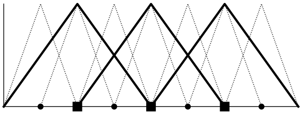
- multilevel subspace decomposition:

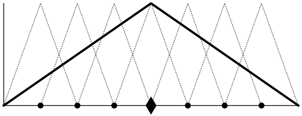
$$\mathcal{V}^h = \mathcal{V}^0 + \mathcal{V}^1 + \cdots + \mathcal{V}^J$$

- hats are combinations of finer hats:

$$\psi_p^{j-1}(\mathbf{x}) = \sum_{q=1}^{m_j} c_{pq} \psi_q^j(\mathbf{x})$$

  - $c_{pq} = \psi_p^{j-1}(\mathbf{x}_q^j)$
  - canonical prolongation P
  - canonical (dual) restriction R
  - see Xu (1992)





NCP RESIDUAL

- consider abstract NCP

$$\begin{aligned} s - b &\geq 0 \\ F(s) &\geq 0 \\ (s-b)F(s) &= 0 \end{aligned}$$

- the solution does not satisfy $F(s) = 0$ everywhere
  - generally $F(s) > 0$ where $s = b$
- how to tell if an iterate $w \approx s$ is converged?
- A. need to monitor NCP residual $\hat{F}(w)$:

$$\hat{F}(w)[\psi_p^J] = \begin{cases} F(w)[\psi_p^J], & w(\mathbf{x}_p^J) > b(\mathbf{x}_p^J) \\ \min\{F(w)[\psi_p^J], 0\}, & w(\mathbf{x}_p^J) = b(\mathbf{x}_p^J) \end{cases}$$

$$\|\hat{F}(w)\|_{(\mathcal{V}^J)'} < \texttt{tol} \quad \implies \quad w \text{ solves NCP}$$

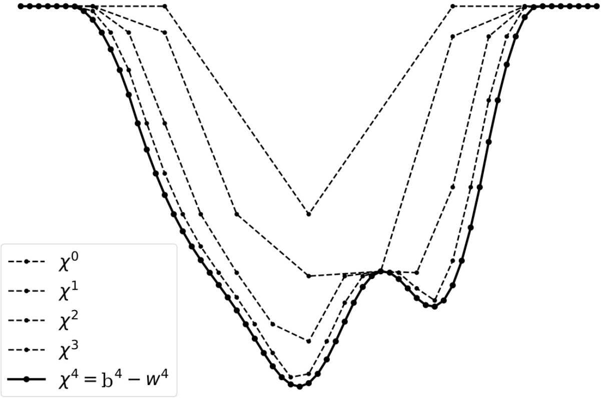
- suppose:
  - b^J is fine-level bed elevation
  - w^J is admissible fine-level iterate ($w^J \geq b^J$)
- define defect constraint and defect constraint set:

$$\chi^J = b^J - w^J$$

$$\mathcal{D}^J = \left\{ v \geq \chi^J \right\} \subset \mathcal{V}^J$$

  - notice $\chi^J \leq 0$
  - $w^J + z^J$ is admissible if and only if $z^J \in \mathcal{D}^J$

- Tai (2003): decompose the fine-level defect constraint onto coarser meshes using monotone restriction



DECOMPOSE THE DEFECT CONSTRAINT

- monotone restriction operator $R^{\oplus} : \mathcal{V}^j \to \mathcal{V}^{j-1}$:

$$R^{\oplus} z = \sum_{p=1}^{m_{j-1}} \max\{z_q \,:\, \psi_p^{j-1}(x_q^j) > 0\}\, \psi_p^{j-1}$$

  - observe $R^{\oplus} z \geq z$
- for $j = 1, \dots, J$ let

$$\chi^{j-1} = R^{\oplus} \chi^j$$

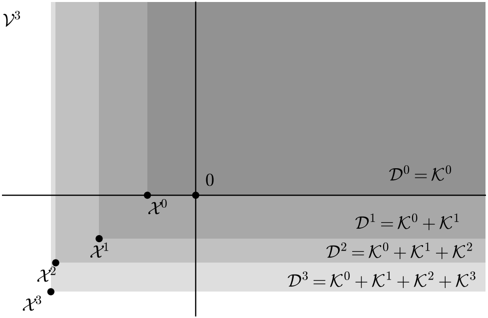
  - also define $\chi^{-1} = 0$
- gaps between defect constraints: $\phi^j = \chi^j - \chi^{j-1}$
- telescoping-sum:

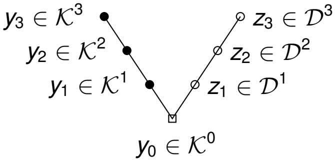
$$\sum_{j=0}^{J} \phi^j = \chi^0 + (\chi^1 - \chi^0) + (\chi^2 - \chi^1) + \cdots + (\chi^J - \chi^{J-1}) = \chi^J$$

- let

$$\mathcal{D}^j = \{v \geq \chi^j\}, \qquad \mathcal{K}^j = \{v \geq \phi^j\}$$

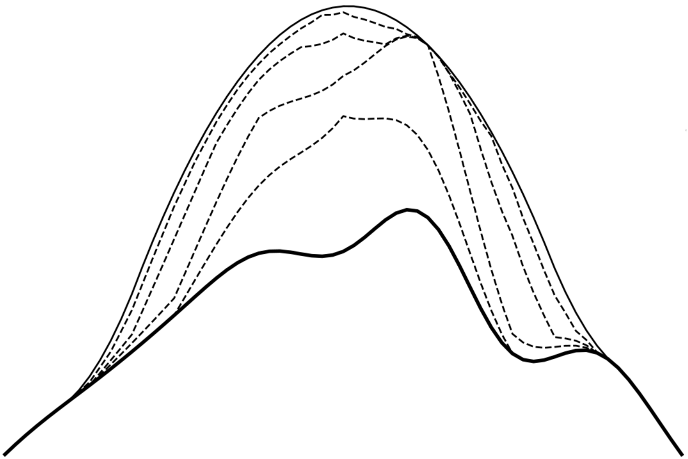
- Tai (2003): decomposition of the fine-level defect constraint set $\mathcal{D}^J$ by cones $\mathcal{D}^j = \mathcal{K}^0 + \cdots + \mathcal{K}^j$ from the inside





MULTILEVEL CONSTRAINT DECOMPOSITION (MCD)

- dishonest attempt to illustrate MCD as decomposition of ice:



FAS-TYPE COARSE CORRECTIONS

- now add an idea not in Tai (2003)
- suppose:
  - g^j is solution iterate on current level
  - down-smoother computes $y^j \in \mathcal{K}^j$
  - new solution iterate is $\tilde{g}^j = g^j + y^j$
- nonlinear coarse correction needs to restrict residual *and* $\tilde{g}^j$
  - full approximation scheme (FAS)
  - in formulas:

$$\begin{aligned} F &= N^j(\tilde{g}^j) - f^j \\ g^{j-1} &= R^\bullet \tilde{g}^j \\ f^{j-1} &= N^{j-1}(g^{j-1}) - RF \end{aligned}$$

where $R^\bullet$ is state restriction (e.g. injection)

NMCD-VCYCLE(J, w^J, χ^J):

  $g^J = w^J$

  for $j = J$ **downto** $j = 1$

    $\chi^{j-1} = R^{\oplus}\chi^j$

    $\phi^j = \chi^j - P\chi^{j-1}$

    $y^j = 0$

    SMOOTHER$^{\text{down}}(j, g^j, y^j, N^j, f^j, \phi^j)$

    $F = N^j(g^j + y^j) - f^j$

    $g^{j-1} = R^{\bullet}(g^j + y^j)$

    $f^{j-1} = N^{j-1}(g^{j-1}) - RF$

  $y^0 = 0$

  SMOOTHER$^{\text{coarse}}(0, g^0, y^0, N^0, f^0, \chi^0)$

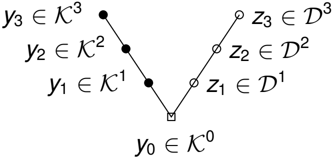
  $z^0 = y^0$

  for $j = 1$ **to** $j = J$

    $z^j = Pz^{j-1} + y^j$

    SMOOTHER$^{\text{up}}(j, g^j, z^j, N^j, f^j, \chi^j)$

  return z^J



↑

up-smoothing corrections
act in larger sets

- SIA residual is local
  - (degenerate) elliptic differential operator
- pointwise smoothers adequate:
  - projected nonlinear Gauss-Seidel (PNGS)
  - projected nonlinear Jacobi (PNJacobi; below)

PNJACOBI($j, g^j, y^j, N^j, f^j, b^j, \texttt{newtonits} = 2, \texttt{omega} = 0.67$):

$\quad$ **for** $k = 1, \dots, \texttt{newtonits}$

$\quad\quad \rho_p(c) := N^j(g^j + y^j + c\psi^j_p)[\psi^j_p] - f^j[\psi^j_p]$

$\quad\quad r_p, \delta_p = \rho_p(0), \rho'_p(0)$

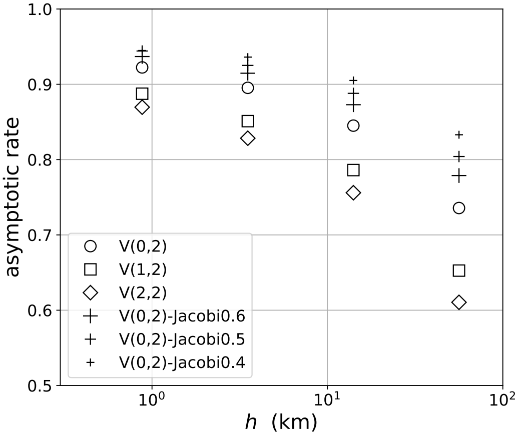
$\quad\quad$ **for** $p = 1, \dots, m_j$

$\quad\quad\quad c = \text{POINTUPDATE}(r_p, \delta_p, y_p, b_p, f^j[\psi^j_p])$

$\quad\quad\quad y_p \leftarrow y_p + \texttt{omega}\, c$

SIA RESULTS

- compare PNGS and PNJacobi smoothers
- GMG v-cycle factors < 1 for SIA
  - up-smoothing preferred, thus $V(0,2)$
  - evidence of mesh independence?



WHAT IS A GOOD STOKES SMOOTHER?

- Stokes residual non-local!
- I have only beginnings of smoother ideas

- goal: GMG for implicit glacier geometry evolution
- much of the tool-chain exists:
    - backward Euler or other stiff scheme
    - NCP or VI for free-boundary problem at each time step
    - nonlinear MCD solution of the VI
        - a form of GMG
- mostly implemented in Firedrake
    - extruded mesh
    - mixed-element, GMG solution of Stokes equations
- outlook for entire approach depends on constructing

  a performant smoother for Stokes dynamics

- I'm kind of stuck, and seeking help!

extra: References

- E. Bueler. Stable finite volume element schemes for the shallow-ice approximation. *J. Glaciol.*, 62 (232):230–242, 2016.
- E. Bueler. Performance analysis of high-resolution ice sheet simulations. arxiv:2206.14352, 2022.
- T. Isaac, G. Stadler, and O. Ghattas. Solution of nonlinear Stokes equations discretized by high-order finite elements on nonconforming and anisotropic meshes, with application to ice sheet dynamics. *SIAM J. Sci. Comput.*, 37(6):B804–B833, 2015.
- G. Jouvet and J. Rappaz. Analysis and finite element approximation of a nonlinear stationary Stokes problem arising in glaciology. *Adv. Numer. Analysis*, 2011.
- B. Kamb and K. A. Echelmeyer. Stress-gradient coupling in glacier flow: I. Longitudinal averaging of the influence of ice thickness and surface slope. *J. Glaciol.*, 32(111):267–284, 1986.
- X.-C. Tai. Rate of convergence for some constraint decomposition methods for nonlinear variational inequalities. *Numer. Math.*, 93(4):755–786, 2003.
- J. Xu. Iterative methods by space decomposition and subspace correction. *SIAM Rev.*, 34:581–613, 1992.

extra: COMPLETE STRONG FORM

- solve one step of backward Euler for $s, \mathbf{u}, p$
- system of NCP coupled to Stokes problem:

$$
\begin{aligned}
s - b &\geq 0 && \text{on } \Omega \\
s - \Delta t\, \mathbf{u}|_s \cdot \mathbf{n}_s - (s_\text{o} + \Delta t\, a) &\geq 0 && \text{"} \\
(s - b)(s - \Delta t\, \mathbf{u}|_s \cdot \mathbf{n}_s - (s_\text{o} + \Delta t\, a)) &= 0 && \text{"} \\
-\nabla \cdot (2\nu_\epsilon(D\mathbf{u})\, D\mathbf{u}) + \nabla p - \rho_\text{i}\mathbf{g} &= \mathbf{0} && \text{on } \Lambda_s \\
\nabla \cdot \mathbf{u} &= 0 && \text{"} \\
\mathbf{u} &= \mathbf{0} && \text{on } \Gamma_0 \\
(2\nu_\epsilon(D\mathbf{u}) D\mathbf{u} - pI)\, \mathbf{n} &= \mathbf{0} && \text{on } \partial\Lambda_s \setminus \Gamma_0
\end{aligned}
$$

  - with regularized Glen-law effective viscosity ($\mathrm{p} = \frac{1}{\mathrm{n}} + 1$):

$$\nu_\epsilon(D\mathbf{u}) = \frac{\Gamma}{2}\left(|D\mathbf{u}|^2 + \epsilon\, D_0^2\right)^{(\mathrm{p}-2)/2}$$