Toward nonlinear multigrid for nonlinear and nonlocal variational inequalities

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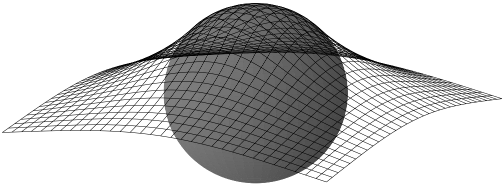
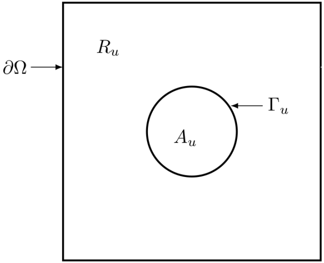
Outline

1. variational inequalities (VIs)
2. full approximation scheme (FAS) multigrid for PDEs
3. the nonlinear and nonlocal VI for a fluid layer in a climate
4. multigrid approaches for VIs
5. FAS multigrid for VIs?

example: a classical obstacle problem



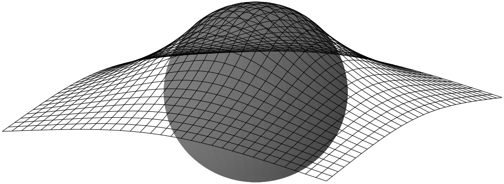


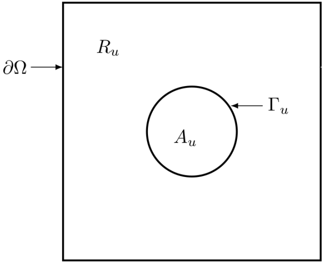
- *problem.* on a domain $\Omega \subset \mathbb{R}^2$, find the displacement $u(x)$ of a membrane, with fixed value $u = g$ on $\partial\Omega$, above an *obstacle* $\psi(x)$, which minimizes the elastic energy

$$J(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 - f\,v$$

- shown above: $\Omega = (-2,2)^2$, $\psi(x)$ a hemisphere, $f(x) = 0$

example: a classical obstacle problem



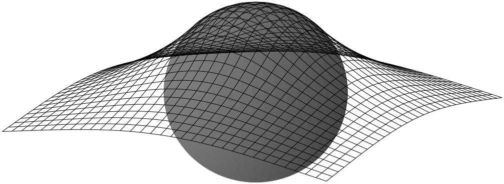


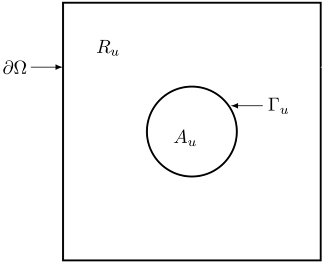
- i.e. constrained optimization over a convex *admissible set*

$$\mathcal{K} = \left\{ v \in H^1(\Omega) \, : \, v\big|_{\partial\Omega} = g \text{ and } v \geq \psi \right\}$$

- $J'(u)$ points directly into $\mathcal{K}$, the *variational inequality* (VI):

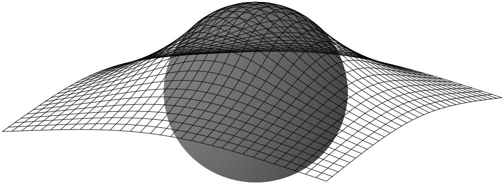
$$\left\langle J'(u), v - u \right\rangle = \int_\Omega \nabla u \cdot \nabla(v - u) - f(v - u) \geq 0 \quad \text{for all } v \in \mathcal{K}$$

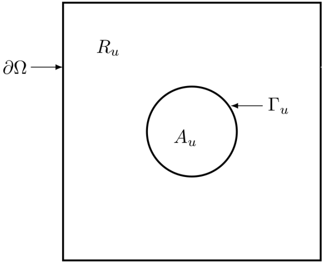




- the solution defines *active* $A_u = \{u = \psi\}$ and *inactive* $R_u = \{u > \psi\}$ subsets of Ω, and a *free boundary* $\Gamma_u = \partial R_u \cap \Omega$
- naive strong form would pose the problem in terms of its solution:

$$\begin{aligned} -\nabla^2 u &= f \quad \text{on } R_u \\ u &= \psi \quad \text{on } A_u \end{aligned}$$





- the *complementarity problem* (CP) is meaningful as a strong form:

$$\begin{aligned} u - \psi &\ge 0 \\ -\nabla^2 u - f &\ge 0 \\ (u - \psi)(-\nabla^2 u - f) &= 0 \end{aligned}$$

  - for optimization problems: CP $=$ KKT conditions

- let $\mathcal{K}$ be a closed and convex subset of a Banach space $\mathcal{V}$
- suppose $F : \mathcal{K} \to \mathcal{V}'$ is a continuous, generally nonlinear operator
  - F may be defined only on $\mathcal{K}$
  - F may *not* be the derivative of an objective function J
- the general problem VI(F,$\mathcal{K}$) is

$$\langle F(u), v - u \rangle \geq 0 \quad \text{for all } v \in \mathcal{K}$$

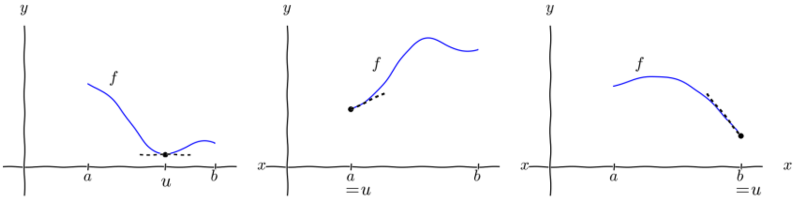
- when $\mathcal{K}$ is nontrivial the problem VI(F,$\mathcal{K}$) is nonlinear *even when F is a linear operator*

VI $=$ "constrained equation"

unconstrained optimization:	constrained optimization:
$\min_{u \in \mathcal{V}} J(u)$	$\min_{u \in \mathcal{K}} J(u)$
equation for $u \in \mathcal{V}$:	VI for $u \in \mathcal{K}$:
$F(u) = 0$	$\langle F(u), v - u \rangle \geq 0 \quad \forall v \in \mathcal{K}$

- elastic contact, Signorini problems (e.g. Kikuchi & Oden 1988)
- viscous contact problems (de Diego et al. 2022)
- pricing of American options in the Black-Scholes model
- the geometry of glaciers $\longleftarrow$ *more soon*

- first-semester calculus



Outline

nonlinear 2-grid scheme

- consider a nonlinear elliptic PDE problem:

$$F(u) = \ell$$

  - for example, $F : \mathcal{V} \to \mathcal{V}'$ for $\mathcal{V} = H^1(\Omega)$, with $\ell \in \mathcal{V}'$
- discretization gives algebraic system on fine grid Ω^h:

$$F^h(u^h) = \ell^h$$

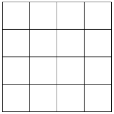
- suppose w^h yields residual norm $\|\ell^h - F^h(w^h)\| > \text{TOL}$

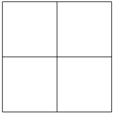
nonlinear 2-grid scheme

- how can we improve w^h *without* globally linearizing F^h? (are there alternatives to Newton's method?)
- note the *residual* $r^h(w^h) = \ell^h - F^h(w^h)$ is computable, while the *error* $e^h = w^h - u^h$ is unknown
- the residual definition can be rewritten

$$F^h(u^h) - F^h(w^h) \overset{*}{=} r^h(w^h)$$

- for F^h linear, try to solve this *error equation* $F^h(e^h) = -r^h(w^h)$ for $\tilde{e}^h$, and correct $w^h \leftarrow w^h - \tilde{e}^h$ to improve w^h?

Ω^h 

 Ω^H

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- *goal*: use a coarser mesh to estimate the error in $*$
- *nodewise problem*: for ψ_i^h a hat function or dof, solve for $c \in \mathbb{R}$:

$$\phi_i(c) = r^h(w^h + c\psi_i^h)[\psi_i^h] = 0$$

- sweeping through and solving nodewise problems is a *smoother*
  - Fourier analysis on linear PDEs shows smoothing property
  - post-smoothing, e^h and $r^h(w^h)$ have smaller high-frequencies
- Brandt (1977): after smoothing, $F^h(u^h) - F^h(w^h) = r^h(w^h)$ should be accurately approximate-able on a coarser grid

nonlinear 2-grid scheme

- *goal*: use a coarser mesh to estimate the error in $*$
- *full approximation storage* (FAS) equation:

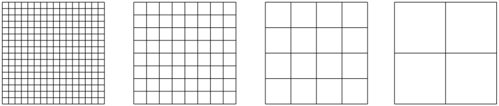
$$F^H(w^H) - F^H(R^\bullet w^h) = R\, r^h(w^h)$$

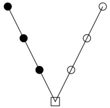
  - $R^\bullet : \mathcal{V}^h \to \mathcal{V}^H$ is *injection*
  - $R : (\mathcal{V}^h)' \to (\mathcal{V}^H)'$ is *canonical restriction*
  - if $w^h = u^h$ exactly then $w^H = R^\bullet w^h$ by well-posedness
- rewritten: $F^H(w^H) = \ell^H$ where $\ell^H = F^H(R^\bullet w^h) + R\, r^h(w^h)$

FAS 2-grid solver

smooth by sweeps over grid:	$w^h \leftarrow [\phi_i(c) = 0\,\forall i]$
restrict:	$\ell^H = F^H(R^\bullet w^h) + R\, r^h(w^h)$
solve coarse:	$F^H(w^H) = \ell^H$
correct :	$w^h \leftarrow w^h + P(w^H - R^\bullet w^h)$
smooth by sweeps over grid:	$w^h \leftarrow [\phi_i(c) = 0\,\forall i]$

- $P : \mathcal{V}^H \to \mathcal{V}^h$ is *prolongation*
- recall: $\phi_i(c) = r^h(w^h + c\psi_i^h)[\psi_i^h]$
- restrict+(solve coarse)+correct $=$ *coarse grid correction*





FAS-VCYCLE$(\ell^J; w^J)$:

for $j = J$ **downto** $j = 1$

SMOOTH$^{\mathrm{down}}(\ell^j; w^j)$

$w^{j-1} \leftarrow R^\bullet w^j$

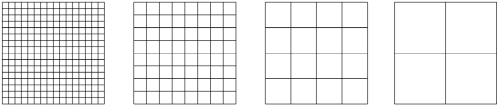
$\ell^{j-1} = F^{j-1}(w^{j-1}) + R\left(\ell^j - F^j(w^j)\right)$

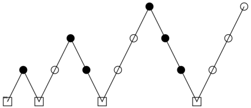
SOLVE$(\ell^0; w^0)$

for $j = 1$ **to** $j = J$

$w^j \leftarrow w^j + P(w^{j-1} - R^\bullet w^j)$

SMOOTH$^{\mathrm{up}}(\ell^j; w^j)$





F-cycle = nested iteration

FAS-VCYCLE$(\ell^J; w^J)$:

- **for** $j = J$ **downto** $j = 1$
  - SMOOTH$^{\mathrm{down}}(\ell^j; w^j)$
  - $w^{j-1} \leftarrow R^\bullet w^j$
  - $\ell^{j-1} = F^{j-1}(w^{j-1}) + R\left(\ell^j - F^j(w^j)\right)$
- SOLVE$(\ell^0; w^0)$
- **for** $j = 1$ **to** $j = J$
  - $w^j \leftarrow w^j + P(w^{j-1} - R^\bullet w^j)$
  - SMOOTH$^{\mathrm{up}}(\ell^j; w^j)$

does it work?

- FAS multigrid works well on the right nonlinear PDE problem
- example: Liouville-Bratu equation[1]

$$-\nabla^2 u - e^u = 0$$

  with Dirichlet boundary conditions on $\Omega = (0,1)^2$
- implement with minimal problem-specific code:
  1. residual evaluation on grid level: $F^j(\cdot)$
  2. pointwise smoother: $\phi_i(c) = 0 \,\forall i$
     - e.g. nonlinear Jacobi or Gauss-Seidel iteration
  3. coarse solve can be same as smoother, or use Newton etc.

[1] exact solution by Liouville (1853) makes a nice test case

multigrid solver composition in PETSc

- implemented here using an FD discretization and PETSc:[2]
  - multigrid solvers in PETSc are *composed* from smoothers on each level, and a coarse-level solver ... here these are nonlinear GS
  - FAS multigrid is a nonlinear solver (SNES) type
  - PETSc $=$ C/Fortran/python
- FAS multigrid F-cycle:

```
./bratu -da_grid_x 5 -da_grid_y 5 -da_refine J \
    -snes_rtol 1.0e-12 \
    -snes_type fas \
    -snes_fas_type full \
    -fas_levels_snes_type ngs \
    -fas_levels_snes_ngs_sweeps 2 \
    -fas_levels_snes_ngs_max_it 1 \
    -fas_levels_snes_norm_schedule none \
    -fas_coarse_snes_type ngs \
    -fas_coarse_snes_max_it 1 \
    -fas_coarse_snes_ngs_sweeps 4 \
    -fas_coarse_snes_ngs_max_it 1
```

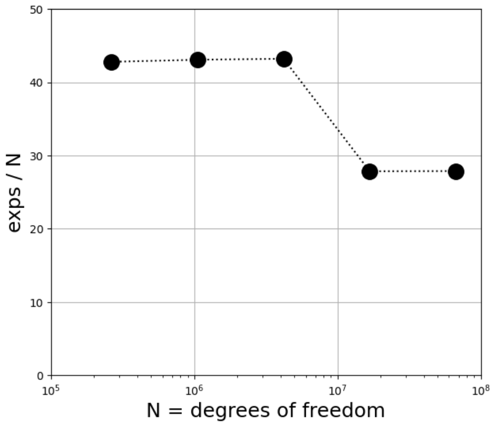
[2] Portable Extensible Toolkit for Scientific computing petsc.org

Bratu model problem: optimality

- observed optimality:

$$\text{flops} = O(N^1)$$
$$\text{exp evaluations} = O(N^1)$$
$$\text{processor time} = O(N^1)$$

- up to $N \approx 10^8$ dofs
  - $J = 11$ refinements
  - laptop is memory-limited
- compare $\approx 20\,\mu\text{s}/\text{N}$ for Poisson equation using Firedrake P_1 elements and geometric multigrid



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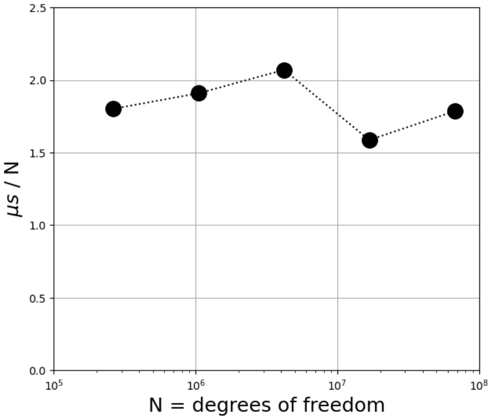
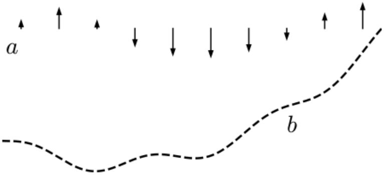


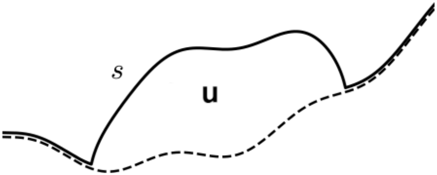
benefits of FAS multigrid for nonlinear PDEs?

- **benefits** of FAS multigrid?
  1. minimal code, esp. in from-scratch implementations
     - just write residual plus pointwise smoother!
  2. composition with nonlinear preconditioners (Brune et al. 2015)
- **disadvantages?**
  1. Firedrake/FENiCs *do* automatically provide linearizations from UFL statements of weak forms
  2. small literature of convergence or descriptive performance for FAS (Trottenberg et al. (2001), Reusken (1987))
  3. not enough tutorial literature?

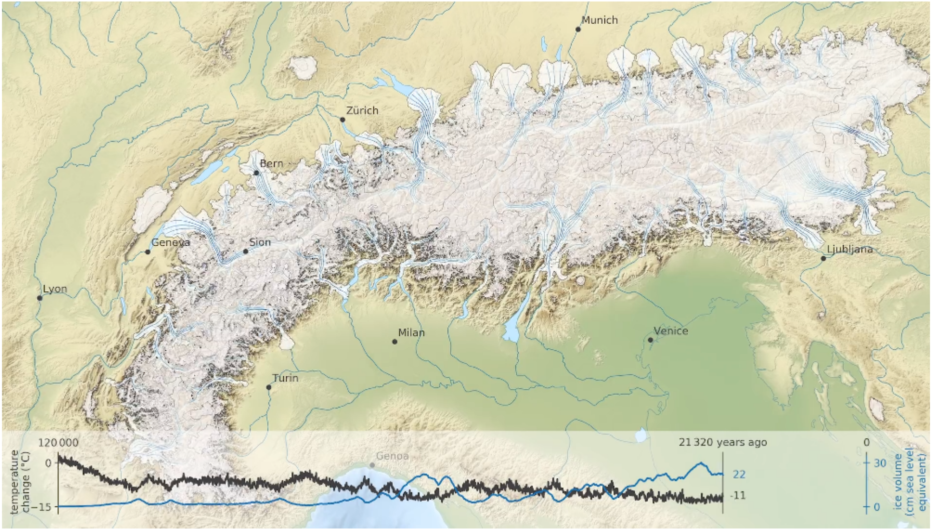
Outline

- let's not get stuck on textbook example problems!
- multigrid for a real-world VI problem?
- consider an incompressible, viscous layer with surface elevation $s(x,y)$, flowing with velocity $\mathbf{u}(x,y,z)$, driven by gravity, over fixed bed topography with elevation $b(x,y)$, in a *climate* which adds or removes fluid at a signed rate $a(x,y)$ $[\mathrm{m\,s^{-1}}]$
    - data a, b defined on domain $\Omega \subset \mathbb{R}^2$
- geophysical examples: **glaciers and ice sheets**, sea ice, lakes



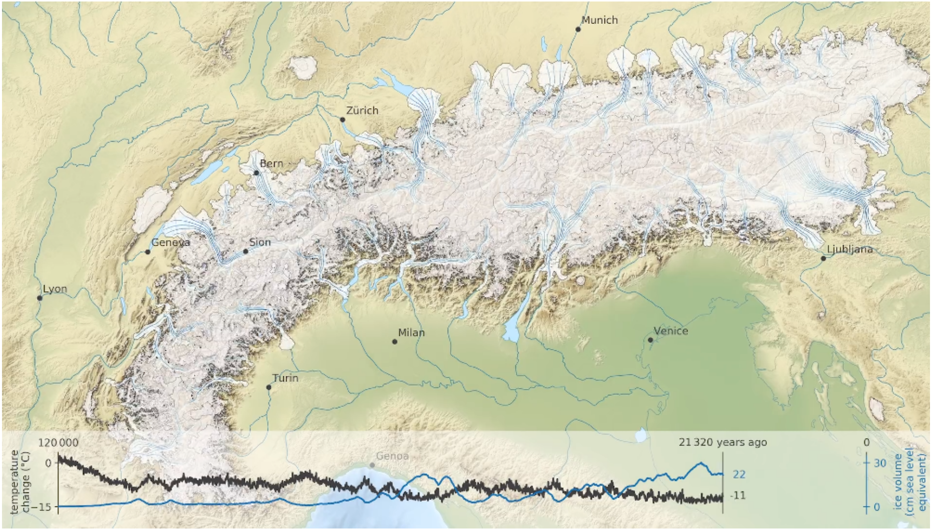


example: glacier ice coverage of the Alps in prior climates



Sequinot et al. (2018)

- more ice sheet modeling at my Math. Geosci. Seminar tomorrow 2pm L5



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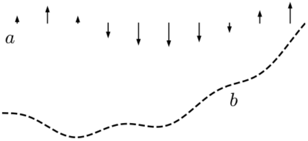
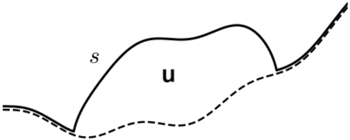
- naive strong form of the steady model:

$$\begin{aligned} s &\geq b && \text{everywhere in } \Omega \\ -\mathbf{u}|_s \cdot \mathbf{n}_s &= a && \text{where } s(x,y) > b(x,y) \end{aligned}$$

  - surface velocity $\mathbf{u}|_s$ is determined by fluid domain geometry s
  - $\mathbf{n}_s = \langle -\nabla s, 1 \rangle$ is upward surface normal
  - generally: $-\mathbf{u}|_s \cdot \mathbf{n}_s$ is a *non-local* function of s
- the inequality constraint $s \geq b$ **generates a free boundary** if an ablative climate $a < 0$ forces surface down to bed





how to evaluate $\Phi(s) = -\mathbf{u}|_s \cdot \mathbf{n}_s$ for glacier ice?

- **Stokes model**
  solve the Stokes problem, then evaluate velocity at surface:

$$\int_{\Lambda(s)=\{b<z<s\}} 2\nu(D\mathbf{u})D\mathbf{u} : D\mathbf{v} - p\nabla\cdot\mathbf{v} - (\nabla\cdot\mathbf{u})q - \rho_{\text{i}}\mathbf{g}\cdot\mathbf{v} = 0 \quad \forall \mathbf{v}, q$$

$$\Phi(s) = -\mathbf{u}|_s \cdot \mathbf{n}_s$$

  - assuming incompressibility and non-Newtonian viscosity: $\nu(D\mathbf{u}) = \frac{1}{2}\Gamma|D\mathbf{u}|^{\text{p}-2}$ with $\text{p} = \frac{4}{3}$
  - given s, this is a well-posed problem for velocity $\mathbf{u} \in \mathbf{W}^{1,\text{p}}$ and pressure $p \in L^{\text{q}}$ on domain $\Lambda(s)$
  - near-optimal solvers available (Isaac et al 2015)

- **lubrication approximation[3] model**
  apply a nonlinear elliptic differential operator to s:

$$\Phi(s) = -\frac{\gamma}{\mathrm{q}}(s-b)^{\mathrm{q}}|\nabla s|^{\mathrm{q}} - \nabla \cdot \left(\frac{\gamma}{\mathrm{q}+1}(s-b)^{\mathrm{q}+1}|\nabla s|^{\mathrm{q}-2}\nabla s \right)$$

  - $\mathrm{q} = 4$
  - ∇ is in x, y only
  - $\Phi(s)$ is a nonlinear **differential operator** in this model because membrane stresses are *not* balanced
  - $\Phi(s)$ is doubly-degenerate

[3] also known as the *shallow ice approximation*

VI for fluid layer in a climate

- admissible surface elevations:

$$\mathcal{K} = \{r \in \mathcal{V} \,:\, r \geq b\}$$

  - $\mathcal{V}$ to be determined by viscous fluid model[4]

- VI problem for surface elevation $s \in \mathcal{K}$:

$$\langle \Phi(s), r - s \rangle \geq \langle a, r - s \rangle \qquad \text{for all } r \in \mathcal{K}$$

where

$$\Phi(s) = -\mathbf{u}|_s \cdot \mathbf{n}_s,$$

with extension by 0 to all of Ω, and $\mathbf{u}$ is the velocity solution on

$$\Lambda(s) = \{(x, y, z) : b(x, y) < z < s(x, y)\}$$

[4] in shallow ice approximation, $(s - b)^{8/3} \in W^{1,4}(\Omega)$ (Jouvet & Bueler, 2012)

CP form of viscous fluid layer in a climate

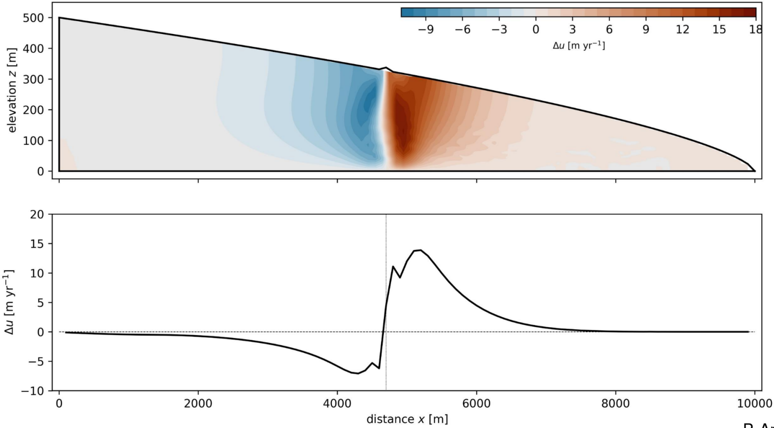
- VI form on previous slide is too abstracted for clarity
- the strong form of the same problem is a **complementarity problem (CP) coupled to a Stokes problem**:

$$\begin{aligned} s - b &\geq 0 && \text{in } \Omega \subset \mathbb{R}^2 \\ -\mathbf{u}|_s \cdot \mathbf{n}_s - a &\geq 0 && \text{"} \\ (s-b)(-\mathbf{u}|_s \cdot \mathbf{n}_s - a) &= 0 && \text{"} \\ -\nabla \cdot (2\nu(D\mathbf{u})\, D\mathbf{u}) + \nabla p - \rho_i \mathbf{g} &= \mathbf{0} && \text{in } \Lambda(s) \subset \mathbb{R}^3 \\ \nabla \cdot \mathbf{u} &= 0 && \text{"} \\ (2\nu(D\mathbf{u})D\mathbf{u} - pI)\,\mathbf{n} &= \mathbf{0} && \{z = s\} \subset \partial\Lambda(s) \\ \mathbf{u} &= \mathbf{0} && \{z = b\} \subset \partial\Lambda(s) \end{aligned}$$

- solve this for s on Ω, and simultaneously for $\mathbf{u}, p$ on $\Lambda(s) = \{b < z < s\}$

a non-local VI problem

- in the Stokes case, the residual $r(s) = a - \Phi(s) = a + \mathbf{u}|_s \cdot \mathbf{n}_s$ **depends non-locally on** s
- for example, consider $\mathbf{u}_{(s+\psi)} - \mathbf{u}_{(s)}$ from surface perturbation (hat function) ψ



P. Arndt figure

what's needed for multigrid to work here?

viscous fluid layer geometry problem

$$\langle \Phi(s), r - s \rangle \geq \langle a, r - s \rangle \qquad \text{for all } r \in \mathcal{K}$$

where

- $\mathcal{K} = \{r \in \mathcal{V} \,:\, r \geq b\}$
- $\Phi(s) = -\mathbf{u}|_s \cdot \mathbf{n}_s$
- s is solution surface elevation
- $\mathbf{u}$ is Stokes solution on $\Lambda(s) = \{b < z < s\}$
- a (climate) and b (bed elevation) are the input data

  - for more on this problem class see (Bueler, 2021)

- what is needed for scalable multilevel solutions?
  1. iterates must be admissible
  2. global linearization of $\Phi(s)$ must be avoided
  3. smoother cost must be comparable to one residual

Outline

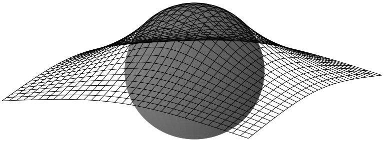
Newton-multigrid for the classical obstacle problem

- VIs are nonlinear problems, even for linear operators like $-\nabla^2$
- Newton-multigrid is straightforward in PETSc:

```
./obstacle -da_grid_x 3 -da_grid_y 3 \
    -snes_type vinewtonrsls -ksp_type cg -pc_type mg \
    -da_refine J
```

  - linear solver applies to inactive variables

    `rsls` = reduced space line search
  - Newton step equations solved by CG with GMG V-cycles
- issue: the outer Newton iteration must converge on the active set **before** multigrid can provide effective preconditioning
  - grid-dependent (growing) Newton iterations



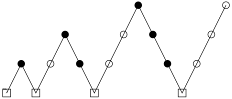
grid	SNES	GMG KSP
17×17	3	4
33×33	6	3
65×65	7	4
129×129	12	4
257×257	21	4

nested iteration

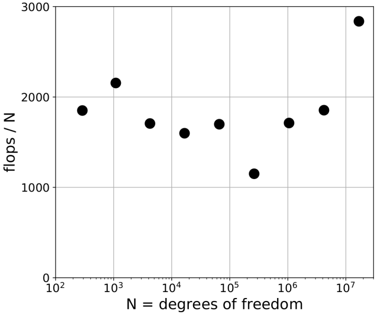
- applying nested iteration (nonlinear F-cycle) resolves this:

```
./obstacle -da_grid_x 3 -da_grid_y 3 \
    -snes_type vinewtonrsls -ksp_type cg -pc_type mg \
    -snes_grid_sequence J
```

  - grid-independent Newton iterations
  - optimal $O(N^1)$ flops and time
  - Chapter 12 example in my new book



grid	SNES	GMG KSP
17×17	3	4
33×33	4	4
65×65	2	4
129×129	2	4
257×257	3	4



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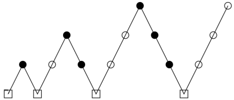

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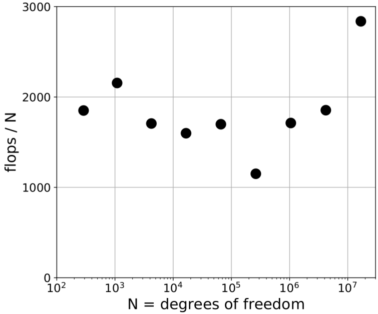

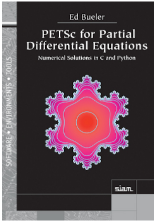

- semi-smooth Newton also yields mesh-independent iterations
  - penalty scaling argument (Farrell et al. 2020)
- other VI multilevel strategies:
  - projected FAS multigrid for linear CPs (Brandt & Cryer, 1983)
  - monotone multigrid (Kornhuber, 1994)
  - multilevel constraint decomposition (Tai, 2003) $\leftarrow$ *more below*

multigrid strategies for VIs: feature table

	admissible iterates	mesh-indep. rates	no global linearization	PETSc or Firedrake
RS NM	✓			✓
+ NI	✓	✓		✓
SS NM		✓		✓
FASCD	✓	**?**	✓	

RS = reduced space, SS = semi-smooth, NM = Newton-multigrid, NI = nested iteration

- for the non-local fluid layer VI problem we need all 4 checked
- we are trying-out a new algorithm,
  FASCD = full approximation storage constraint decomposition
    - Firedrake implementation

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FASCD	✓	**?**	✓	✓

RS = reduced space, SS = semi-smooth, NM = Newton-multigrid, NI = nested iteration

- for the non-local fluid layer VI problem we need all 4 checked
- we are trying-out a new algorithm,
  FASCD = full approximation storage constraint decomposition
  - Firedrake implementation . . . as of yesterday

Outline

constraint decomposition

- Tai's (2003) *constraint decomposition* (CD) for VIs follows the subspace decomposition idea (Xu 1992)
- suppose $\mathcal{K} \subset \mathcal{V}$ is a closed and convex admissible subset
- for a subspace decomposition $\mathcal{V} = \sum_i \mathcal{V}_i$, write the admissible subset as a sum

$$\mathcal{K} = \sum_i \mathcal{K}_i$$

  where $\mathcal{K}^i \subset \mathcal{V}^i$, with projections $\Pi_i : \mathcal{K} \to \mathcal{K}_i$
- CD additive and multiplicative iterations exist for $VI(F, \ell, \mathcal{K})$

CD-ADD(u):
  for $i \in \{0, \dots, m-1\}$:
    find $\hat{w}_i \in \mathcal{K}_i$ so that for all $v_i \in \mathcal{K}_i$,
      $\boxed{\langle F(u - \Pi_i u + \hat{w}_i), v_i - \hat{w}_i \rangle \geq \langle \ell, v_i - \hat{w}_i \rangle}$
  $\hat{w} = \sum_i \hat{w}_i \in \mathcal{K}$
  return $w = (1 - \alpha) u + \alpha \hat{w}$

multilevel constraint decomposition

- recall $\mathcal{K} = \{v \geq \psi\}$ in classical obstacle problem
- define **defect obstacle** for a fine-level iterate w^J:

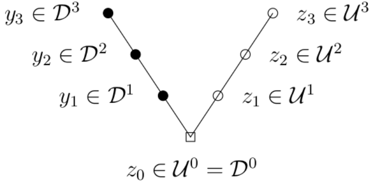
$$\chi^J = \psi^J - w^J$$

- **monotone restriction** generates obstacles on each level:

$$\chi^j = R^{\oplus} \chi^{j+1}$$

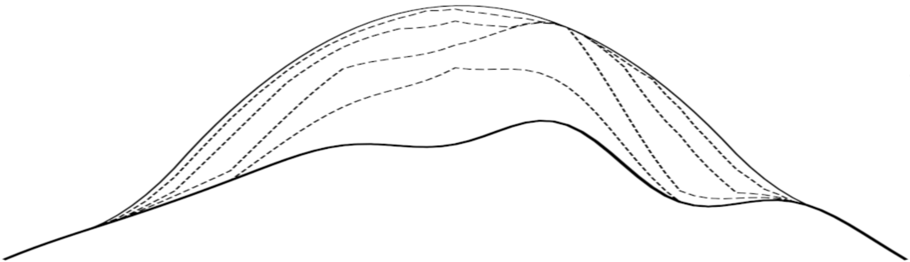
- let $\mathcal{U}^j = \{z \geq \chi^j\}$, $\mathcal{D}^j = \{y \geq \chi^j - \chi^{j-1}\}$
- get CD of fine-level constraint set:

$$\mathcal{U}^J = \sum_{i=0}^{J} \mathcal{D}^i$$



- multiplicative CD iteration $\rightarrow$ V-cycle

- again this is too abstract
- what does it look like for the fluid layer?
    - coarse grids have *admissible* pieces of the fine-grid iterate



full approximation storage constraint decomposition

$\textsc{fascd-vcycle}(\ell^J, \psi^J; w^J)$:

$\chi^J = \psi^J - w^J$

for $j = J$ **downto** $j = 1$

$\chi^{j-1} = R^{\oplus}\chi^j$

$\phi^j = \chi^j - P\chi^{j-1}$

$y^j = 0$

$\textsc{smooth}^{\text{down}}(\ell^j, \phi^j, w^j; y^j)$ (*smoothing in* $\mathcal{D}^j$)

$w^{j-1} = R^{\bullet}(w^j + y^j)$

$\ell^{j-1} = f^{j-1}(w^{j-1}) + R\left(\ell^j - f^j(w^j + y^j)\right)$

$z^0 = 0$

$\textsc{solve}(\ell^0, \chi^0, w^0; z^0)$ (*coarse solve in* $\mathcal{U}^0$)

for $j = 1$ **to** $j = J$

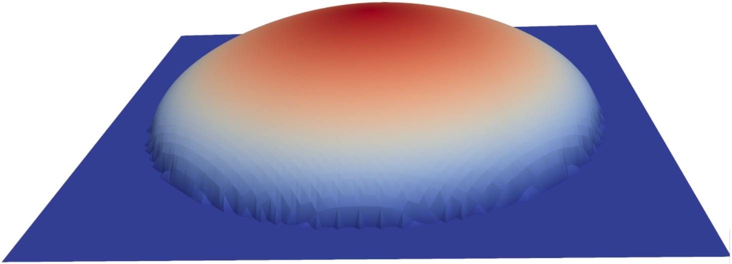
$z^j = y^j + Pz^{j-1}$

$\textsc{smooth}^{\text{up}}(\ell^j, \chi^j, w^j; z^j)$ (*smoothing in* $\mathcal{U}^j$)

$w^J \leftarrow w^J + z^J$

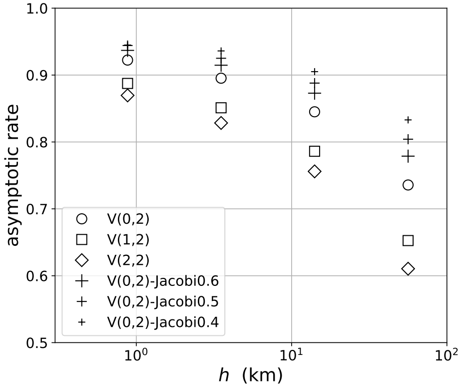
2D shallow ice approximation results (*very fresh*)

- preliminary results
- dome test case in lubrication approximation
  - here $\Phi(s)$ *is* a differential operator
  - note $s^{8/3} \in W^{1,4}(\Omega)$ but not in C^2
- FASCD algorithm result
  - Firedrake P_1 elements
  - strong smoother (`vinewtonrsls`)



evidence of mesh independence

- same lubrication approximation, but in 1D
- FASCD V-cycles with NGS and NJacobi smoothers
- up-smoothing preferred: V(0,2) beats V(1,1)
- evidence of mesh independence of factors $\|r^{(k+1)}\|/\|r^{(k)}\|$



summary and outlook

- the **variational inequality (VI)** problem class is good to know
- likewise **full approximation storage (FAS) multigrid**
  - need for better support and documentation in PETSc/Firedrake
- multigrid treatment of nonlinear and **nonlocal VIs**?
  - smoothers not obvious in nonlocal cases
  - seeking practical evidence of mesh-independent convergence
- glacier evolution, as fluid-layer-in-climate problems, **needs attention** from applied mathematicians and numerical analysts
  - VI form not widely recognized
  - current state of the art = explicit time stepping of surface
    - ▷ slow for science, intrinsically *not* scalable
  - to do: **steady-state and implicit step VI problems**
  - more on this view in my Math. Geosci. seminar tomorrow 2pm L5

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