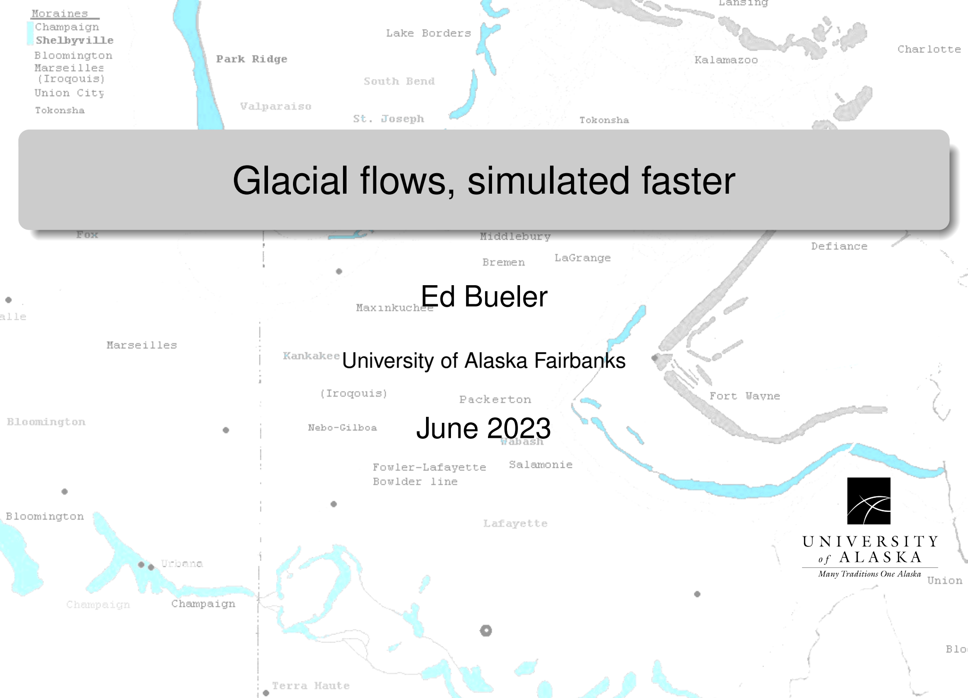


Glacial flows, simulated faster

Ed Bueler

University of Alaska Fairbanks

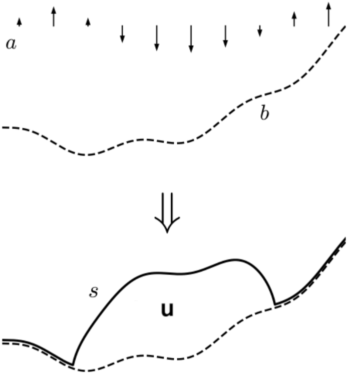
June 2023

UNIVERSITY of ALASKA
Many Traditions One Alaska

Outline

basic facts about glaciers

- glacier ice is a *very viscous, incompressible, non-Newtonian fluid*
  - more soon . . .
- glaciers lie on *topography*
  - except sometimes they float on water (floating tongue or ice shelf)
- a glacier's geometry (*free surface*), and its velocity, *evolve in contact with the climate*:
  - snowfall
  - surface melt
  - subglacial melt
  - sub-shelf melt (when floating)
  - calving (into ocean)

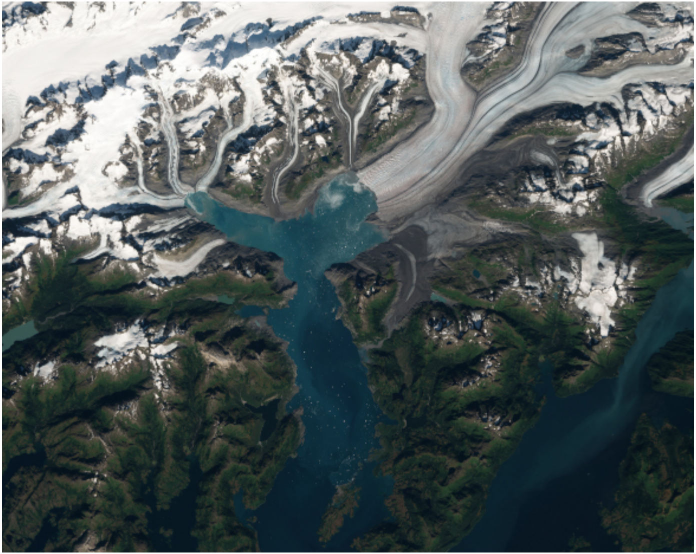


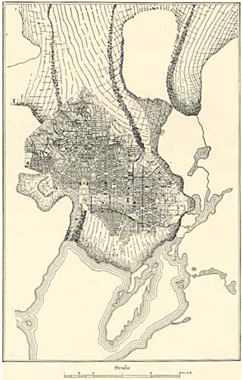
Polaris Glacier (Post and LaChappelle 1971)

pictures of glaciers

Taku Glacier (M. Truffer 2016)

pictures of glaciers



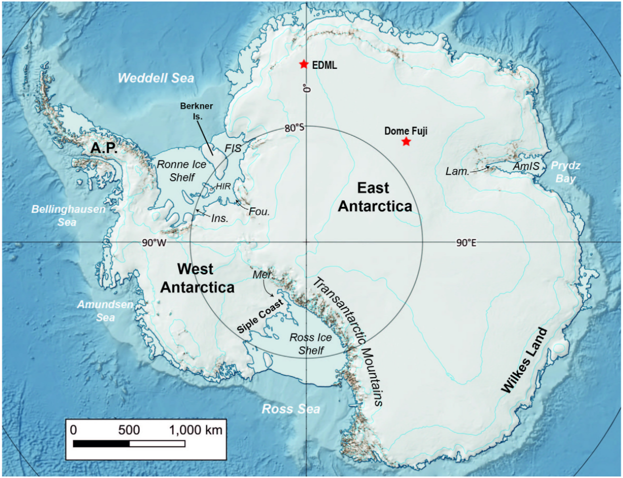


Columbia Glacier

(Sentinel-2B 2018, National Geographic 1910)

what is an ice sheet?

- *def.* **ice sheet** = a large glacier with small thickness/width ratio



Antarctic ice sheet (Pittard et al 2021)

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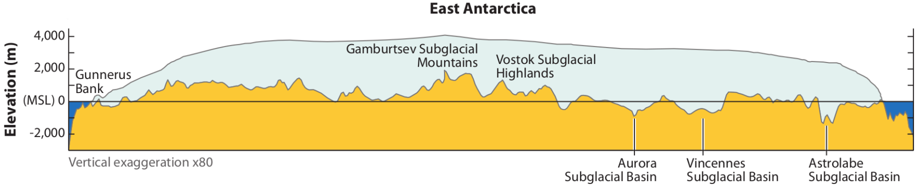

note smooth surface and rough bed . . . and vertical exaggeration (Schoof & Hewitt 2013)

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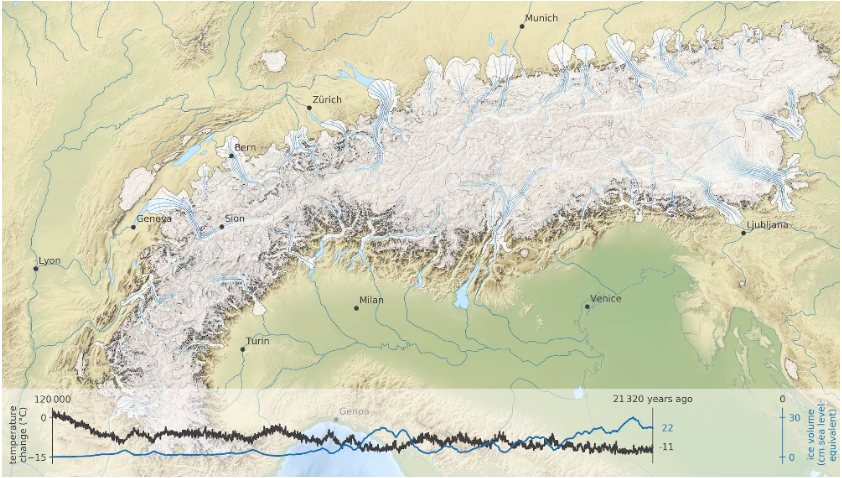

modeled Alpine ice sheet near last glacial maximum (Seguinot et al 2018)

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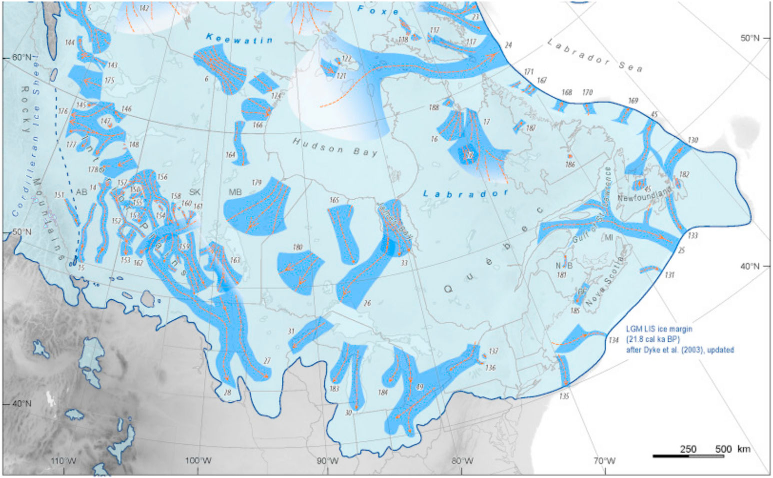

Laurentide ice sheet, ≈ 22,000 years ago (Margold, Stokes, Clark 2018)

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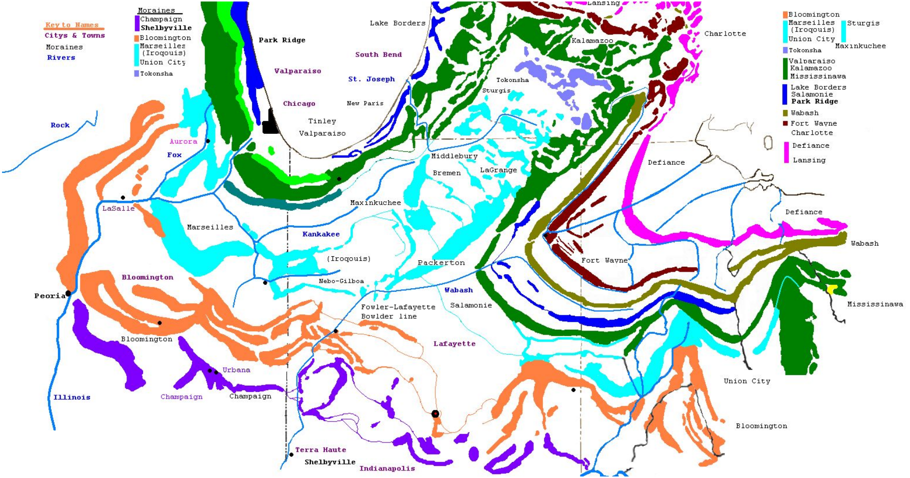
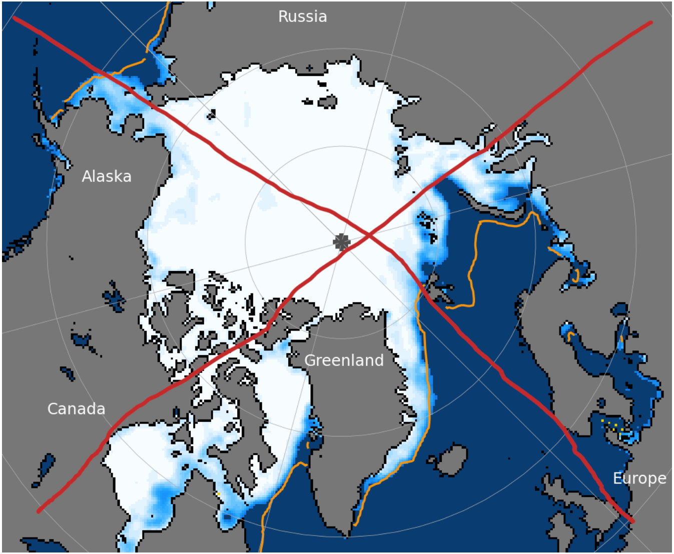


moraines in Illinois, Indiana, Ohio (Larsen 1986 and other sources)

finally, an ice sheet is *not* sea ice!



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basic facts about glaciers . . . again

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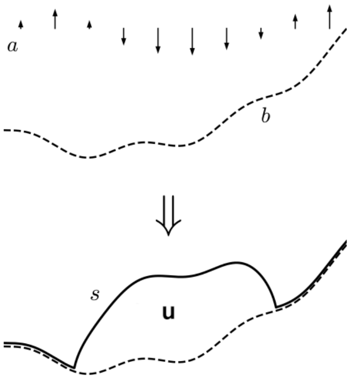


modeling simplifications

- for simplicity/clarity of the upcoming model, I will **ignore** these aspects of glacier physics in my talk:
  - floating ice
  - subglacial hydrology
  - ice temperature
  - fracture processes (e.g. calving)
  - solid earth deformation
- all are important for doing science!
- UAF's Parallel Ice Sheet Model (`pism.io`), for example, includes these and other processes

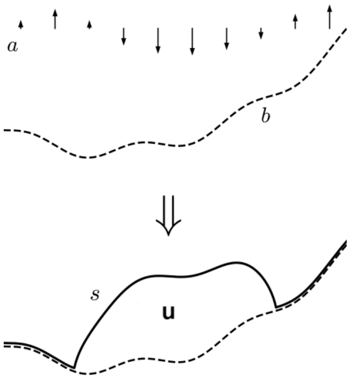
Definition

a **glacier model** is a map
which evolves a glacier in a climate

- at least two inputs:
  - *surface mass balance*

$$a(t,x,y) = \begin{pmatrix}\text{snowfall minus} \\ \text{melt \& runoff}\end{pmatrix}$$

    - units of mass flux: $\text{kg}\,\text{m}^{-2}\text{s}^{-1}$
  - *bed elevation* $b(x,y)$
- at least two outputs:
  - *upper surface elevation* $s(t,x,y)$
  - *ice velocity* $\mathbf{u}(t,x,y,z)$
- map: $\begin{pmatrix}\text{climate \&} \\ \text{topography}\end{pmatrix} \to \begin{pmatrix}\text{geometry} \\ \text{\& velocity}\end{pmatrix}$



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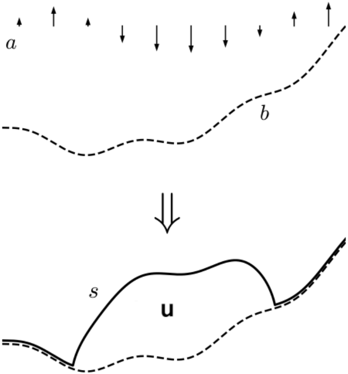


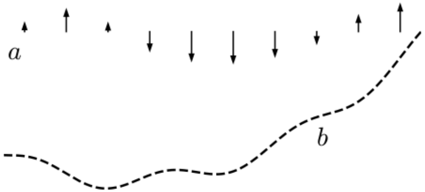
- data $a(t,x,y)$, $b(x,y)$ are defined on a **fixed domain**:

$$t \in [0,T] \quad \text{and} \quad (x,y) \in \Omega \subset \mathbb{R}^2$$

- solution **surface elevation** $s(t,x,y)$ is defined on $[0,T] \times \Omega$
  - also a fixed domain,
  - but $s = b$ where there is no ice
- $s(t,x,y)$ determines the **icy domain** $\Lambda(t) \subset \mathbb{R}^3$:

$$\Lambda(t) = \{(x,y,z)\,:\,b(x,y) < z < s(t,x,y)\}$$

- the solution **velocity** $\mathbf{u}(t,x,y,z)$ is defined on $\Lambda(t)$



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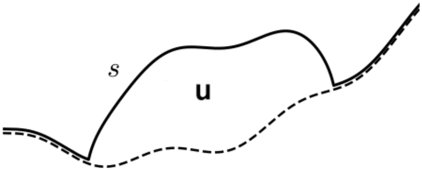

the basic glacier model: conservation

- glacier evolution is merely physics . . . so it **conserves**
  - mass
  - momentum
  - energy ← *ignored for simplicity in this talk*
- conservation of mass happens
  - within the icy domain $\Lambda(t) \subset \mathbb{R}^3$:

$$\textbf{incompressibility} \qquad \nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Lambda(t)$$

  - on the surfaces $\Gamma_s(t), \Gamma_b(t) \subset \partial\Lambda(t)$:

$$\textbf{surface kinematic equation (SKE)} \qquad \frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s = a \qquad \text{on } \Gamma_s(t)$$

$$\textbf{non-penetration} \qquad \mathbf{u}|_b \cdot \mathbf{n}_b = 0 \qquad \text{on } \Gamma_b(t)$$

    - ▷ $\Gamma_s(t)$ is upper surface of the ice
    - ▷ $\Gamma_b(t)$ is base of the ice
    - ▷ $\mathbf{n}_s = \langle -\nabla s, 1 \rangle$ is upward surface normal

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the free boundary problem for a fluid layer

- glacier evolution is a **free-boundary** problem
- specifically, the surface kinematic equation (SKE)

$$\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s = a$$

  applies *only* on the ice upper surface $\Gamma_s(t)$
- in the remainder of the (fixed) domain $\Omega \subset \mathbb{R}^2$, **complementarity** holds:

$$s = b \quad \text{and} \quad a \le 0$$

- for more on this perspective see Bueler (2021), *Conservation laws for free-boundary fluid layers*, SIAM J. Appl. Math

- **nonlinear complementarity problem** (NCP) :

$$
\begin{aligned}
s - b &\geq 0 && \text{on } \Omega \subset \mathbb{R}^2 \\
\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a &\geq 0 && \text{''} \\
(s-b)\left(\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a\right) &= 0 && \text{''} \\
-\nabla \cdot (2\nu(D\mathbf{u})\, D\mathbf{u}) + \nabla p - \rho_i \mathbf{g} &= \mathbf{0} && \text{in } \Lambda(t) \subset \mathbb{R}^3 \\
\nabla \cdot \mathbf{u} &= 0 && \text{''} \\
\tau_b - \mathbf{f}(\mathbf{u}|_b) &= \mathbf{0} && \text{on } \Gamma_b(t) \\
\mathbf{u}|_b \cdot \mathbf{n}_b &= 0 && \text{''} \\
(2\nu(D\mathbf{u})D\mathbf{u} - pI)\, \mathbf{n}_s &= \mathbf{0} && \text{on } \Gamma_s(t)
\end{aligned}
$$

  - note: $\mathbf{u}|_s = 0$ where no ice
  - viscosity by Glen law: $2\nu(D\mathbf{u}) = \Gamma |D\mathbf{u}|^{p-2}$, $p \approx 4$

- nonlinear complementarity problem (NCP) coupled to **Stokes**:

$$
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the basic glacier model is a DAE system

- for this slide, forget complementarity and boundary conditions to get simplified model "SKE coupled to Stokes":

$$\begin{aligned}\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a &= 0 \\ -\nabla \cdot (2\nu(D\mathbf{u})\, D\mathbf{u}) + \nabla p - \rho_i \mathbf{g} &= \mathbf{0} \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

- only the first of these 5 equations has a time derivative
  - recall: ice is very viscous and incompressible
- this time-dependent problem is a **differential algebraic equation** (DAE), an extremely stiff system:

$$\begin{aligned}\dot{x} &= f(x, y) \\ 0 &= g(x, y)\end{aligned}$$

  - in ∞ dimensions, of course,
  - and also subject to complementarity

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- to the best of my knowledge, *no* current research groups are studying well-posedness or regularity for this basic model
  - though most researchers would agree NCP-coupled-to-Stokes is indeed the intended model!
- progress has been made on well-posedness of the lubrication approximation of the basic model, the so-called **shallow ice approximation**:
  - 1D well-posedness on flat bed (Calvo et al 2002)
  - 2D steady-state existence on general beds (Jouvet & Bueler 2012)
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the basic glacier model: current numerical *thinking*

- numerical glacier and ice sheet modelers tend to think of the Stokes problem separately from surface evolution
  - *time-splitting* or *explicit time-stepping* is often taken for granted
- ...and ice sheet geometry evolution is often addressed with minimal awareness of complementarity
- the NCP-coupled-to-Stokes basic model is *not yet* in common use for high-resolution, long-duration ice sheet simulations
  - because it is too slow
  - **can we make it fast enough to use?** ← *what I am working on*

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Outline

the mass-continuity equation view

- "thickness transport form" helps for evolution or stability questions
- define:

$H(t,x,y) = s - b$	ice thickness
$\mathbf{U}(t,x,y) = \frac{1}{H}\int_b^s \mathbf{u}\,dz$	vertically-averaged horizontal velocity

  - note s and H are equivalent variables for modeling ice geometry
- the **mass continuity equation** for thickness, an apparent **advection equation**, follows from the SKE and incompressibility:

$$\frac{\partial H}{\partial t} + \nabla \cdot (\mathbf{U} H) = a$$

- *question:* is this really an advection equation?
  answer: not really . . . ice flows (mostly) downhill so

$$\mathbf{U} \sim -\nabla s \sim -\nabla H$$

- in any case, the NCP-coupled-to-Stokes system *has no characteristic curves*

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mass continuity equation: advection or diffusion?

advective schema: $$\frac{\partial H}{\partial t} + \nabla \cdot (\mathbf{U} H) = a$$

diffusion schema: $$\frac{\partial H}{\partial t} - \nabla \cdot (D \nabla s) = a$$

- both forms are nonlinear: $\mathbf{U} = \mathbf{U}(H, \nabla s),\ D = D(H, \nabla s)$
- the glacier modeling literature is confusing!
- the diffusion schema is literal in the shallow ice approximation
  - more on this momentarily
- regardless of your schema preference, the fact that ice flows downhill has *time-stepping stability* consequences!
- …so let us recall some traditional numerical analysis

traditional PDE time-stepping

advective schema: $$\frac{\partial H}{\partial t} + \nabla \cdot (\mathbf{U} H) = a$$

diffusion schema: $$\frac{\partial H}{\partial t} - \nabla \cdot (D \nabla s) = a$$

- **explicit** time stepping is common for **advections**
- for example, forward Euler using spacing h and time step Δt:

$$\frac{H_j^{\ell+1} - H_j^{\ell}}{\Delta t} + \frac{\mathbf{q}_{j+1/2}^{\ell} - \mathbf{q}_{j-1/2}^{\ell}}{h} = a_j^{\ell}$$

  - need good approximations of flux $\mathbf{q} = \mathbf{U} H$: upwinding, Lax-Wendroff, streamline diffusion, flux-limiters, ...
  - conditionally stable, with CFL maximum time step

$$\Delta t \le \frac{h}{\max |\mathbf{U}|} = O(h)$$

traditional PDE time-stepping

advective schema: $$\frac{\partial H}{\partial t} + \nabla \cdot (\mathbf{U} H) = a$$

diffusion schema: $$\frac{\partial H}{\partial t} - \nabla \cdot (D \nabla s) = a$$

- **explicit** time stepping for **diffusions** is best avoided
- for example, forward Euler:

$$\frac{H_j^{\ell+1} - H_j^\ell}{\Delta t} - \frac{D_{j+\frac{1}{2}}(s_{j+1}^\ell + s_j^\ell) - D_{j-\frac{1}{2}}(s_j^\ell + s_{j-1}^\ell)}{h^2} = a_j^\ell$$

  - conditionally stable, with maximum time step

$$\Delta t \le \frac{h^2}{\max D} = O(h^2)$$

traditional PDE time-stepping

advective schema: $$\frac{\partial H}{\partial t} + \nabla \cdot (\mathbf{U} H) = a$$

diffusion schema: $$\frac{\partial H}{\partial t} - \nabla \cdot (D \nabla s) = a$$

- **implicit** time stepping for **diffusions** is often recommended
- for example, backward Euler:

$$\frac{H_j^{\ell+1} - H_j^{\ell}}{\Delta t} - \frac{D_{j+\frac{1}{2}}(s_{j+1}^{\ell+1} + s_j^{\ell+1}) - D_{j-\frac{1}{2}}(s_j^{\ell+1} + s_{j-1}^{\ell+1})}{h^2} = a_j^{\ell}$$

  - unconditionally stable, but must solve equations at each step
  - further implicit schemes: Crank-Nicolson, BDF, ...

time-stepping in current ice sheet models

- current-technology, large-scale numerical models, including PISM, use explicit time stepping
  - this is embarrassing: the mathematical problem is a DAE
- many researchers "believe" the advection schema
  - time step is supposed to be determined by CFL using the coupled solution velocity $\mathbf{U}$
- the accuracy/performance/usability consequences of the suppressed DAE/diffusive character are hard to sweep under the rug
- the whole situation is a cry for mathematical clarity!

time-stepping in *future* ice sheet models

- **implicit time-stepping** is appropriate for DAE problems
- future models will solve a sequence of NCP-coupled-to-Stokes free-boundary problems at each time step

Outline

1. introduction to glaciers and ice sheets
2. the basic mathematical model for glaciers
3. numerics: time-stepping
4. numerics: Stokes models
5. numerics: comparative performance analysis
6. a multilevel approach
7. conclusion

- the simplest of glacier flow approximations is the "lubrication" approximation: **shallow ice approximation** (SIA)
- SIA version of the NCP:

$$s - b \geq 0, \quad \frac{\partial s}{\partial t} + \Phi(s) - a \geq 0, \quad (s-b)\left(\frac{\partial s}{\partial t} + \Phi(s) - a\right) = 0$$

the surface motion contribution $\Phi(s) = -\mathbf{u}|_s \cdot \mathbf{n}_s$ has a formula:

$$\Phi(s) = -\frac{\gamma}{\mathrm{p}}(s-b)^{\mathrm{p}}|\nabla s|^{\mathrm{p}} - \nabla \cdot \left(\frac{\gamma}{\mathrm{p}+1}(s-b)^{\mathrm{p}+1}|\nabla s|^{\mathrm{p}-2}\nabla s\right)$$

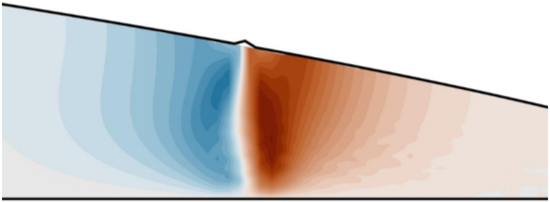
  - constants $\mathrm{p} = \mathrm{n} + 1$ and $\gamma > 0$ relate to ice deformation

- $\Phi(s)$ is a **doubly-nonlinear differential operator**
  - porous medium and p-Laplacian type simultaneously
  - but *local* in surface and bed topography, which Stokes is not
  - well-posedness holds for the weak form = **variational inequality** (Calvo et al 2002, Jouvet & Bueler 2012, Piersanti & Temam 2022)

- however, from now on, let us avoid shallowness approximations
- the basic glacier model (NCP coupled to Stokes) problem has a **non-local** surface velocity function $\Phi(s) = -\mathbf{u}|_s \cdot \mathbf{n}_s$

$$s - b \geq 0, \quad \frac{\partial s}{\partial t} + \Phi(s) - a \geq 0, \quad (s - b)\left(\frac{\partial s}{\partial t} + \Phi(s) - a\right) = 0$$

- the Stokes velocity solution responds to a surface perturbation by up- and down-stream changes, for several ice thicknesses, while the SIA velocity responds only underneath the surface perturbation



- however, from now on, let us avoid shallowness approximations
- the basic glacier model (NCP coupled to Stokes) problem has a **non-local** surface velocity function $\Phi(s) = -\mathbf{u}|_s \cdot \mathbf{n}_s$

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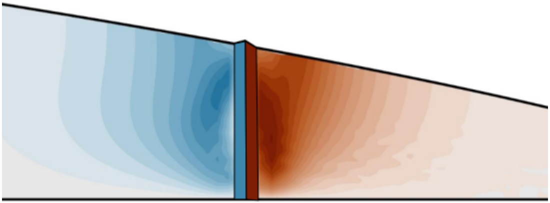

- a non-shallow model solves a Stokes problem at each step:

$$\begin{aligned} -\nabla \cdot (2\nu(D\mathbf{u})\, D\mathbf{u}) + \nabla p - \rho_\text{i}\mathbf{g} &= \mathbf{0} && \text{in } \Lambda \subset \mathbb{R}^3 \\ \nabla \cdot \mathbf{u} &= 0 && \text{''} \\ \tau_b - \mathbf{f}(\mathbf{u}|_b) &= \mathbf{0} && \text{on } \Gamma_b \\ \mathbf{u}|_b \cdot \mathbf{n}_b &= 0 && \text{''} \\ (2\nu(D\mathbf{u}) D\mathbf{u} - pI)\, \mathbf{n}_s &= \mathbf{0} && \text{on } \Gamma_s \end{aligned}$$

- this is the **stress balance** (conservation of momentum) problem which determines velocity **u** and pressure p
- how fast is the numerical solution process?
  - how do solution algorithms **scale** with increasing spatial resolution?

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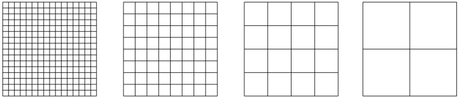
\end{aligned}$$

- this is the **stress balance** (conservation of momentum) problem which determines velocity **u** and pressure *p*
- how fast is the numerical solution process?
  - how do solution algorithms **scale** with increasing spatial resolution?

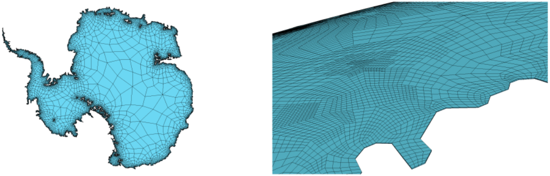
- consider the Poisson equation:

$$-\nabla^2 u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

- discretization generates a linear system $A\mathbf{u} = \mathbf{b}$ with $\mathbf{u} \in \mathbb{R}^m$
- the number of unknowns is the data size m:
  - m = #(nodes in the mesh)
  - m scales with mesh cell diameter: $m \sim h^{-2}$ in 2D
- **complexity** or **algorithmic scaling** of flops, as $m \to \infty$, depends on solver algorithm:
  - $O(m^3)$ for direct linear algebra, ignoring matrix structure
  - $\approx O(m^2)$ for sparsity-exploiting direct linear algebra
  - $O(m^1)$, **optimal**, for **multigrid** solvers



- Stokes: m = #(velocity and pressure unknowns)
- model the scaling as $O(m^{1+\alpha})$, with $\alpha = 0$ optimal
- **near-optimal solvers** already exist: ← *good news!*
  - $\alpha = 0.08$ for Isaac et al. (2015) Stokes solver
    - ▷ unstructured quadrilateral/tetrahedral mesh, $Q_k \times Q_{k-2}$ stable elements, Schur-preconditioned Newton-Krylov, ice-column-oriented algebraic multigrid (AMG) preconditioner for $(\mathbf{u}, \mathbf{u})$ block

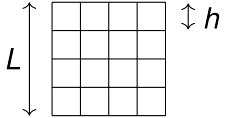


  - $\alpha = 0.05$ for Tuminaro et al (2016) 1st-order (shallow) AMG solver
  - similar for Brown et al (2013) 1st-order (shallow) GMG solver
- *but* this is for Stokes solvers **de-coupled** from the surface-evolution NCP

Outline

- ice sheets are thin layers, thus ice sheet models often have $O(1)$ mesh points in the vertical direction
  - e.g. Issac et al (2015) Stokes solver
  - I am ignoring refinement in the vertical
- data size: $m = \#$(surface elevation & velocity & pressure unknowns)
- assume domain $\Omega \subset \mathbb{R}^2$ with width L and cell diameter h:

$$m \sim \frac{L^2}{h^2}$$



- recall explicit time-stepping stability:

advective	$\frac{\partial H}{\partial t} + \nabla \cdot (\mathbf{U} H) = a$	$\Longrightarrow$	$\Delta t \leq \frac{h}{U}$
diffusion	$\frac{\partial H}{\partial t} - \nabla \cdot (D \nabla s) = a$	$\Longrightarrow$	$\Delta t \leq \frac{h^2}{D}$

- recall stress-balance solver complexity: $O(m^{1+\alpha})$

- glaciologists want to run time-stepping high-resolution simulations of ice sheets over e.g. 10^5 year ice age cycles
- proposed metric: **flops per model year**
- the question:

  how does this metric **scale** in the **high spatial resolution limit** $h \to 0$, equivalently $m \to \infty$?

- the goal is optimality: flops $\sim O(h^{-2}) = O(m^1)$

ice sheet models: explicit time-stepping performance

time-stepping		*flops per model year*
explicit	SIA	$O\left(\dfrac{D\,L^2}{h^4}\right) = O\left(\dfrac{D}{L^2}m^2\right)$
explicit (*advective*)	Stokes	$O\left(\dfrac{U\,L^{2+2\alpha}}{h^{3+2\alpha}}\right) = O\left(\dfrac{U}{L}m^{1.5+\alpha}\right)$
(*diffusive*)	Stokes	$O\left(\dfrac{D\,L^{2+2\alpha}}{h^{4+2\alpha}}\right) = O\left(\dfrac{D}{L^2}m^{2+\alpha}\right)$

- we *want* optimality: $O(m^1)$ flops per model year
- explicit time-stepping implies **too many stress-balance solves**
  - while the Stokes (stress-balance) scaling exponent α is important, even Stokes solver optimality ($\alpha = 0$) cannot yield optimality

- let us try **implicit time-stepping**, for its unconditional stability
- each step is now a **free-boundary NCP-coupled-to-Stokes problem**
- let us parameterize cost of these solves as $O(m^{1+\beta})$
- we still need q model updates per year to integrate climate influences, and track evolution for the simulation purpose

time-stepping		*flops per model year*
explicit	SIA	$O\left(\frac{D\,L^2}{h^4}\right) = O\left(\frac{D}{L^2} m^2\right)$
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(*diffusive*)	Stokes	$O\left(\frac{D\,L^{2+2\alpha}}{h^{4+2\alpha}}\right) = O\left(\frac{D}{L^2} m^{2+\alpha}\right)$
implicit		$O\left(\frac{q\,L^{2+2\beta}}{h^{2+2\beta}}\right) = O\left(q\,m^{1+\beta}\right)$

- new goal: use implicit time-stepping *and* build a $\beta \approx 0$ NCP-coupled-to-Stokes solver for problem at each time step

existing implicit models?

- no convincing NCP-coupled-to-Stokes (free-boundary) solvers exist yet
  - however, Wirbel & Jarosch (2020) is an important beginning
- the Bueler (2016) implicit and NCP SIA solver scales badly: $\beta = 0.8$

Outline

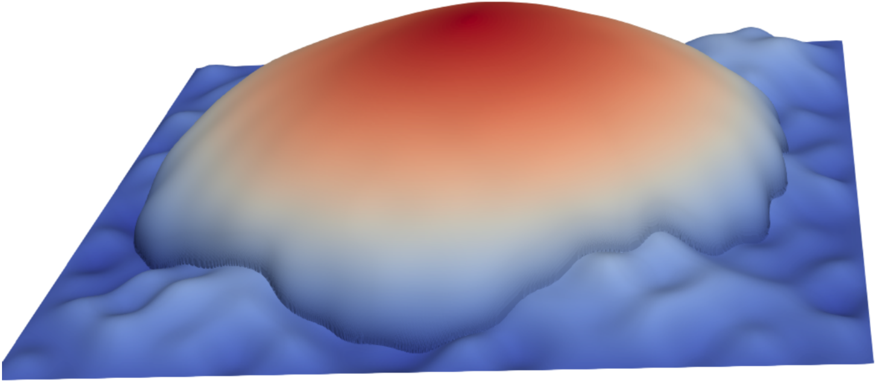
- direct attack on the NCP-coupled-to-Stokes problem, to get an optimal ($\beta = 0$) solver, seems to require a **multilevel** solver for **variational inequalities** (VIs)
- but in the **non-local residual** case
  - application of the smoother needs to reduce the NCP residual from the surface-motion term $\Phi(s) = -\mathbf{u}|_s \cdot \mathbf{n}_s$, where $\mathbf{u}|_s$ is evaluated from a scalable Stokes solver
- this seems not to exist, but we are making progress ...

a *new* multilevel SIA solver (joint with P. Farrell)

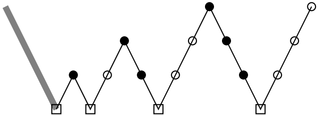
FASCD

full approximation storage constraint decomposition
a multilevel method for box-constrained NCPs and VIs

- in preparation, but here are fresh preliminary results . . .



- results below show FASCD F-cycles give optimal ($\beta = 0$) performance for the SIA NCP problem
  - *iterations* = number of V-cycles after F-cycle "ramp"
  - *time* is for 4-core runs on my laptop



levels	*m*	*iterations*	*time* (s)
2	20^2	5	3.10
3	40^2	4	3.55
4	80^2	4	4.39
5	160^2	4	7.12
6	320^2	4	17.66
7	640^2	5	69.92
8	1280^2	5	284.02
9	2560^2	4	1006.41

Outline

summary

- glacier simulations are both **important to humanity** and a rich **source of interesting mathematics**
  - predict sea level rise!
- ice sheet models solve a multi-scale, irregular-data problem with hard-to-observe boundary conditions
  - there are **no easy or magic techniques** for performance
- current-technology ice sheet models mostly use **explicit** time stepping, **non-optimal** stress-balance solvers, and **shallow** assumptions
  - progress is being made in all of these areas, e.g. scalable Stokes solvers (Isaac et al. 2015)
- scalable solvers for implicit-step, NCP-coupled-to-Stokes models require **multilevel solvers for non-local variational inequalities**
  - is this the preferred numerical design for the basic glacier model?

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