Fast solvers for partial differential equations subject to inequalities

Ed Bueler [1] Patrick Farrell [2]

[1] University of Alaska Fairbanks

[2] Mathematical Institute, Oxford University

November 2023

Outline: *Fast solvers for PDEs subject to inequalities*

1. variational inequalities (VIs)
2. nonlinear multigrid for PDEs
3. multigrid for VIs
4. results

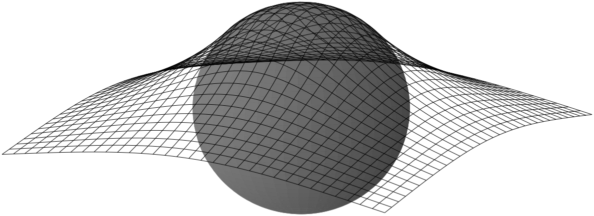
MATH 692 Finite Element Seminar in Spring 2024 (Thursdays 3:30–4:30pm)

Outline: *Fast solvers for PDEs subject to inequalities*

1. variational inequalities (VIs)
2. nonlinear multigrid for PDEs
3. multigrid for VIs
4. results

MATH 692 Finite Element Seminar in Spring 2024 (Thursdays 3:30–4:30pm)

example: classical obstacle problem

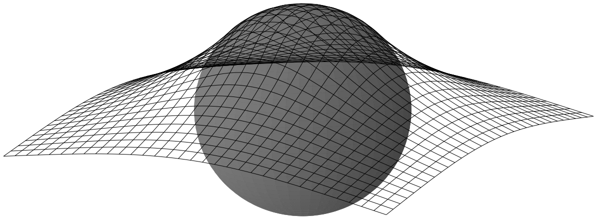


- *problem.* on a domain $\Omega \subset \mathbb{R}^2$, find the displacement $u(x)$ of a membrane, with fixed value $u = g$ on $\partial\Omega$, above an *obstacle* $\psi(x)$, which minimizes elastic (plus some potential) energy

$$J(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 - f\,v$$

- shown above: Ω a square, $\psi(x)$ a hemisphere

Q. how to solve this as a PDE with boundary conditions?



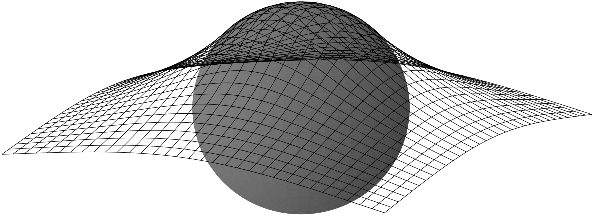
- *problem.* on a domain $\Omega \subset \mathbb{R}^2$, find the displacement $u(x)$ of a membrane, with fixed value $u = g$ on $\partial\Omega$, above an *obstacle* $\psi(x)$, which minimizes elastic (plus some potential) energy

$$J(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 - f\,v$$

- shown above: Ω a square, $\psi(x)$ a hemisphere

Q. how to solve this as a PDE with boundary conditions?

example: classical obstacle problem



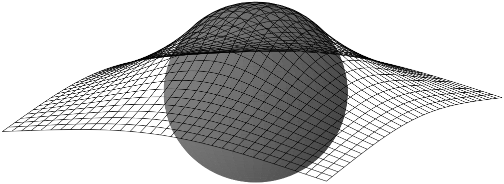
- this is constrained optimization over an infinite-dimensional *admissible set*

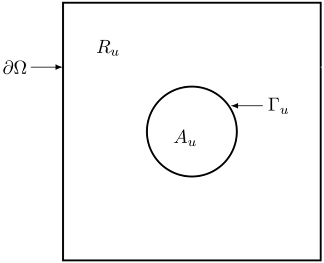
$$\mathcal{K} = \left\{ v \in H^1(\Omega) \,:\, v\big|_{\partial\Omega} = g \text{ and } v \geq \psi \right\}$$

  - $\mathcal{K}$ is a closed and convex subset of the Sobolev space

$$H^1(\Omega) = \left\{ v \,:\, \int_\Omega |v|^2 + |\nabla v|^2 < \infty \right\}$$

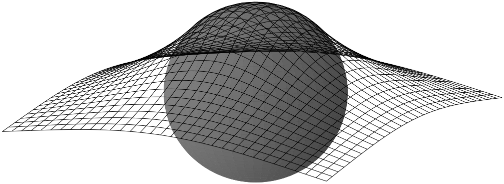
example: classical obstacle problem

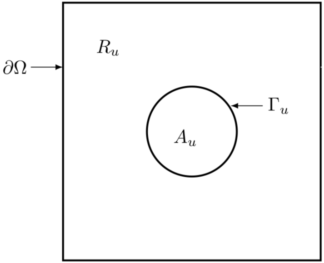




- the solution defines subsets of Ω:
  - *active set* $A_u = \{u = \psi\}$
  - *inactive set* $R_u = \{u > \psi\}$
  - *free boundary* $\Gamma_u = \partial R_u \cap \Omega$

example: classical obstacle problem



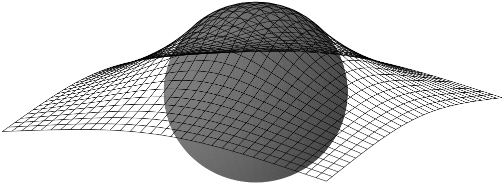


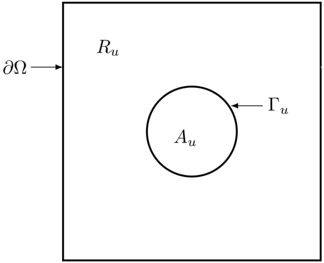
- a naive strong form would pose the problem in terms of its solution:

$$\begin{aligned} -\nabla^2 u &= f \quad \text{on } R_u \\ u &= \psi \quad \text{on } A_u \end{aligned}$$

  - Poisson equation $-\nabla^2 u = f$ is "$J'(u) = 0$" on R_u
  - using the solution u to define the set R_u on which to solve the PDE $-\nabla^2 u = f$ **does not lead to solution algorithms**

example: classical obstacle problem



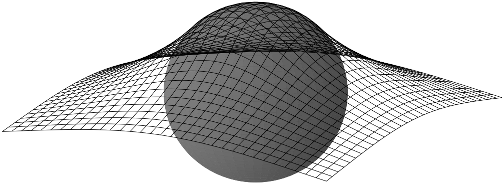


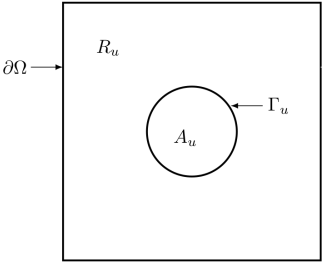
- the *complementarity problem* (CP) is a meaningful strong form:

$$\begin{aligned} u - \psi &\geq 0 \\ -\nabla^2 u - f &\geq 0 \\ (u - \psi)(-\nabla^2 u - f) &= 0 \end{aligned}$$

  - CP $=$ KKT conditions
    - but in ∞-dimensions

example: classical obstacle problem





- the weak form is a *variational inequality* (VI), which says that $J'(u)$ points directly into $\mathcal{K}$:

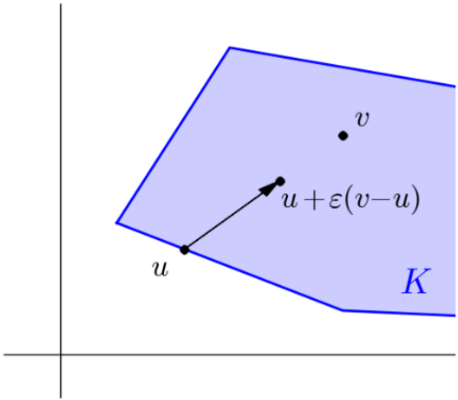
$$\langle J'(u), v-u \rangle = \int_\Omega \nabla u \cdot \nabla(v-u) - f(v-u) \geq 0$$

for all $v \in \mathcal{K}$

VI $=$ weak form

- for problems of optimization type, the VI is the weak form, with $v - u$ as the test function:

$$J(u) \le J(v) \quad \forall v \in \mathcal{K} \qquad \iff \qquad \langle J'(u), v - u \rangle \ge 0 \quad \forall v \in \mathcal{K}$$



- let $\mathcal{K}$ be a closed and convex subset of a Banach space $\mathcal{V}$
- suppose $F : \mathcal{K} \to \mathcal{V}'$ is a continuous operator
    - F is generally nonlinear
    - F may be defined *only* on $\mathcal{K}$
    - F may *not* be the derivative of an objective function J
    - $F = J'$, a linear operator, in classical obstacle problem
- the general variational inequality VI(F,$\mathcal{K}$) is

$$\langle F(u), v - u \rangle \geq 0 \quad \text{for all } v \in \mathcal{K}$$

- when $\mathcal{K}$ is nontrivial the problem VI(F,$\mathcal{K}$) is nonlinear *even when F is a linear operator*

VI $=$ constrained "system of equations"

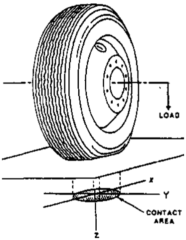
	unconstrained	constrained
optimization	$\min_{u \in \mathcal{V}} J(u)$	$\min_{u \in \mathcal{K} \subset \mathcal{V}} J(u)$
equations	find $u \in \mathcal{V}$: $F(u) = 0$	find $u \in \mathcal{K} \subset \mathcal{V}$: $\langle F(u), v - u \rangle \geq 0 \quad \forall v \in \mathcal{K}$

VI $=$ constrained "system of equations"

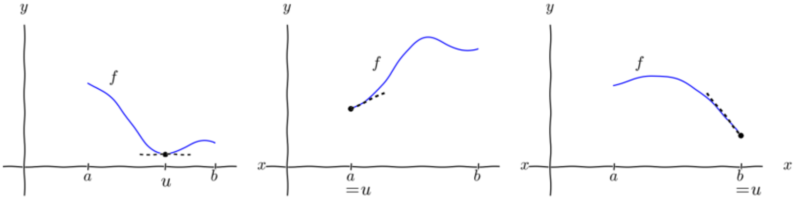
	unconstrained	constrained
optimization	$\min_{u\in\mathcal{V}} J(u)$	$\min_{u\in\mathcal{K}\subset\mathcal{V}} J(u)$
weak form equations	find $u \in \mathcal{V}$: $\langle F(u), v\rangle = 0 \quad \forall v \in \mathcal{V}$	find $u \in \mathcal{K} \subset \mathcal{V}$: $\langle F(u), v - u\rangle \geq 0 \quad \forall v \in \mathcal{K}$

applications of VIs

- elastic contact
  - car tires, for example
- pricing of American options
  - inequality-constrained Black-Scholes model
- the geometry of glaciers
- first-semester calculus:



$$u \leftarrow \min_{x \in [a,b]} f(x) \qquad \Longleftrightarrow \qquad f'(u)(v-u) \geq 0 \quad \forall v \in [a,b]$$



Outline

1. variational inequalities (VIs)
2. nonlinear multigrid for PDEs
   - full approximation scheme (FAS)
3. multigrid for VIs
   - full approximation scheme constraint decomposition (FASCD)
4. results
   - classical obstacle problem
   - advection-diffusion of a concentration
   - glacier surface elevations

nonlinear 2-mesh scheme

- consider a nonlinear elliptic PDE problem:

$$F(u) = \ell$$

  - $u \in \mathcal{V} = H^1(\Omega)$
  - $\ell \in \mathcal{V}'$
  - $F : \mathcal{V} \to \mathcal{V}'$ continuous and one-to-one
  - for example, the Liouville-Bratu problem: $-\nabla^2 u - e^u = f$
- discretization gives algebraic system on fine mesh Ω^h:

$$F^h(u^h) = \ell^h$$

  - u^h denotes exact (algebraic) solution

Ω^h  Ω^H

- *goal*: to solve $F^h(u^h) = \ell^h$ on Ω^h
- suppose w^h is a not-yet-converged iterate:

$$r^h = \ell^h - F^h(w^h), \qquad \|r^h\| > \text{TOL}$$

- how can we improve w^h *without* globally linearizing F^h?
  - are there alternatives to Newton's method?
- notes:
  i) the *residual* $\quad r^h = \ell^h - F^h(w^h) \quad$ is computable
  ii) the *error* $\quad e^h = u^h - w^h \quad$ is unknown
  iii) our equation can be rewritten

$$F^h(u^h) - F^h(w^h) = r^h$$

nonlinear 2-mesh scheme

- *updated goal*: from iterate w^h, to solve

$$F^h(u^h) - F^h(w^h) = r^h$$

- **for F^h linear**, convert this to the *error equation*

$$F^h(e^h) = r^h$$

- an approximation solution $\tilde{e}^h$ would improve our iterate:

$$w^h \leftarrow w^h + \tilde{e}^h$$

- but F^h is not linear!

- *updated goal*: from iterate w^h, to solve

$$F^h(u^h) - F^h(w^h) = r^h$$

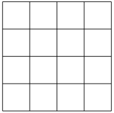
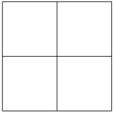
- **for F^h linear**, convert this to the *error equation*

$$F^h(e^h) = r^h$$

- an approximation solution $\tilde{e}^h$ would improve our iterate:

$$w^h \leftarrow w^h + \tilde{e}^h$$

- but F^h is not linear!

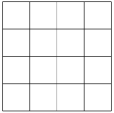
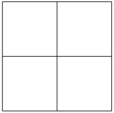
Ω^h   Ω^H

- *updated goal*: use a coarser mesh Ω^H to somehow estimate the solution u^h in the nonlinear *correction equation*

$$F^h(u^h) - F^h(w^h) = r^h$$

- basic multigrid idea: there are algorithms (**smoothers**) which "improve" w^h ... use them a little first ... then correct from the coarser mesh
  - "improve" means they remove high-frequency error components efficiently

nonlinear 2-mesh scheme

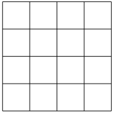
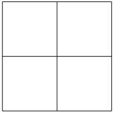
Ω^h   Ω^H

- *nodewise problem*: for ψ_i^h a hat function or dof, solve for $c \in \mathbb{R}$ to make the residual at that location zero:

$$\phi_i(c) = r^h(w^h + c\psi_i^h)[\psi_i^h] = 0$$

- sweeping through and solving nodewise problems is a **smoother**
  - Fourier analysis shows smoothing property
  - after smoothing, e^h and r^h have smaller high-frequencies
- after smoothing, the correction equation on Ω^h should be accurately approximate-able on the coarser mesh Ω^H

nonlinear 2-mesh scheme

Ω^h   Ω^H

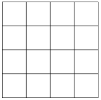
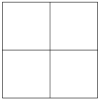
- *updated goal*: use a coarser mesh Ω^H to somehow estimate the solution u^h in $F^h(u^h) - F^h(w^h) = r^h(w^h)$
- Brandt's (1977) *full approximation scheme* (FAS) equation:

$$F^H(u^H) - F^H(R^\bullet w^h) = R\, r^h(w^h)$$

  - $R^\bullet : \mathcal{V}^h \to \mathcal{V}^H$ is node-wise *injection*
  - $R : (\mathcal{V}^h)' \to (\mathcal{V}^H)'$ is *canonical restriction*
  - note: if $w^h = u^h$ exactly then $u^H = R^\bullet w^h$ since F^H injective
- rewritten FAS equation: let $\ell^H = F^H(R^\bullet w^h) + R\, r^h(w^h)$ then

$$F^H(u^H) = \ell^H$$

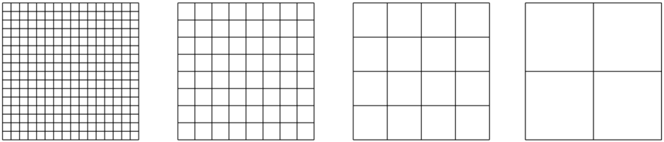
full approximation scheme (FAS) 2-mesh solver

fine mesh $= \Omega^h$   $\Omega^H =$ coarse mesh

pre-smooth over fine:	[smoother updates w^h]
restrict:	$\ell^H = F^H(R^\bullet w^h) + R\, r^h(w^h)$
solve coarse:	$F^H(w^H) = \ell^H$
correct:	$w^h \leftarrow w^h + P(w^H - R^\bullet w^h)$
post-smooth over fine:	[smoother updates w^h]

- $P : \mathcal{V}^H \to \mathcal{V}^h$ is *canonical prolongation*
- **restrict**+(**solve coarse**)+**correct** $=$ *FAS coarse grid correction*

nonlinear multigrid by FAS: V-cycle



$J = 3$ $\quad j = 2$ $\quad j = 1$ $\quad j = 0$

FAS-VCYCLE$(\ell^J; w^J)$:

for $j = J$ **downto** $j = 1$

$\quad$ SMOOTH$^{\mathrm{down}}(\ell^j; w^j)$

$\quad$ $w^{j-1} = R^\bullet w^j$

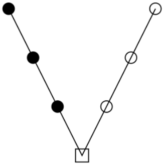
$\quad$ $\ell^{j-1} = F^{j-1}(w^{j-1}) + R\left(\ell^j - F^j(w^j)\right)$

SOLVE$(\ell^0; w^0)$

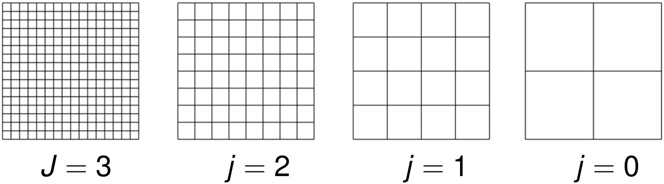
for $j = 1$ **to** $j = J$

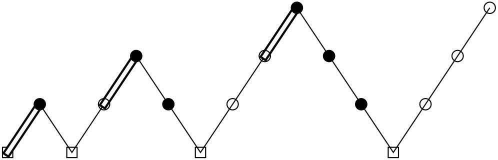
$\quad$ $w^j \leftarrow w^j + P(w^{j-1} - R^\bullet w^j)$

$\quad$ SMOOTH$^{\mathrm{up}}(\ell^j; w^j)$



nonlinear multigrid by FAS: FMG cycle





FMG $=$ full multigrid

- FAS multigrid works **very well** on nice nonlinear PDE problems
- example: Liouville-Bratu equation

$$-\nabla^2 u - e^u = 0$$

  with Dirichlet boundary conditions on $\Omega = (0,1)^2$
- discretize by (straightforward) finite differences
- minimal problem-specific code:
  1. residual evaluation on grid level: $F^j(\cdot)$
  2. pointwise smoother: $\phi_i(c) = 0 \,\forall i$
     - nonlinear Gauss-Seidel iteration
  3. coarsest-level solve can be same as smoother, or more sophisticated (e.g. Newton iteration)

- what does "very well" on the previous slide mean?

definition

a solver is *optimal* if work in flops, and/or run-time, is $O(N)$ for N unknowns

- since $\sim$1980: optimality can be achieved by multigrid for PDE problems with reasonably-smooth solutions
- in fact, multigrid people get greedy

definition

a solver shows *textbook multigrid efficiency* if it does total work less than 10 times that of a single smoother sweep

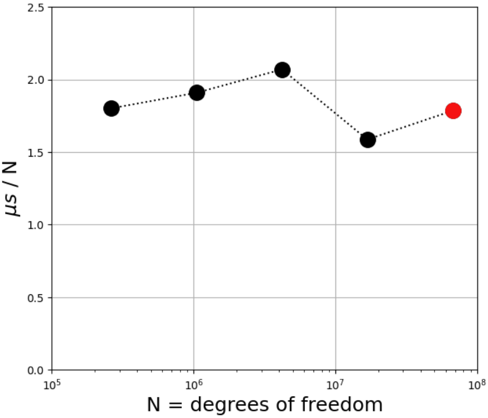
  - TME $\implies$ optimal

Bratu model problem: TME

- `bratu.c`
- observed optimality:

$$\text{flops} = O(N^1)$$
$$\text{processor time} = O(N^1)$$

- highest-resolution 12-level V-cycle has $N \approx 10^8$ unknowns
- compare $\approx 20\,\mu\text{s}/\text{N}$ for Poisson using Firedrake (P_1, geometric multigrid)

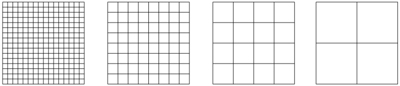


Outline

an FAS multigrid strategy for VIs

- new algorithm (Bueler & Farrell 2023):
  FASCD = full approximation scheme constraint decomposition
- what is "constraint decomposition" in FAS<u>CD</u>?

subspace decomposition



Ω^3 Ω^2 Ω^1 Ω^0

- start with subspace decomposition over nested meshes:

$$\Omega^j \subset \Omega^{j+1}$$

- the FE function spaces $\mathcal{V}^j$ over Ω^j are also nested:

$$\mathcal{V}^j \subset \mathcal{V}^{j+1}$$

definition

$$\mathcal{V}^J = \sum_{i=0}^{J} \mathcal{V}^i$$ is called a *subspace decomposition* (Xu 1992)

- ○ *non*-unique vector space sum
- ○ Xu's paper explains how to analyze linear multigrid for PDEs via subspace decomposition

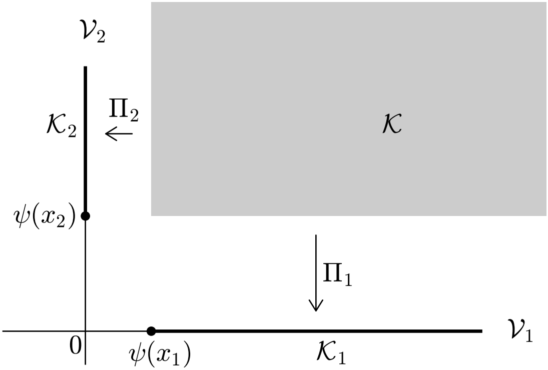
- Tai's (2003) constraint decomposition *non-trivially* extends a subspace decomposition $\mathcal{V}^J = \sum_i \mathcal{V}^i$ to convex subsets
- suppose $\mathcal{K}^J \subset \mathcal{V}^J$ is a closed and convex subset

definition

$\mathcal{K}^J = \sum_{i=0}^{J} \mathcal{K}^i$ is a *constraint decomposition* (CD) if there are closed and convex subsets $\mathcal{K}^i \subset \mathcal{V}^i$, and (nonlinear) projections $\Pi_i : \mathcal{K}^J \to \mathcal{K}^i$, so that $v = \sum_{i=0}^{J} \Pi_i v$ and a stability condition applies (not shown)

constraint decomposition

- observation: generally $\mathcal{K}^i \not\subset \mathcal{K}^J$



obstacle problem on a two-point mesh with $\mathcal{V} \cong \mathbb{R}^2$

iterations over constraint decomposition

- Tai proposed abstract iterations for solving $VI(F, \ell, \mathcal{K})$ over a CD $\mathcal{K}^J = \sum_{i=0}^{J} \mathcal{K}^i$

CD-MULT(u):
  for $i = 0, \dots, m-1$**:**
    find $w_i \in \mathcal{K}_i$ **s.t.**

$$\left\langle F\Big(\sum_{j<i} w_j + w_i + \sum_{j>i} \Pi_j u\Big),\, v_i - w_i \right\rangle \geq \langle \ell, v_i - w_i \rangle \;\forall v_i \in \mathcal{K}_i$$

  return $w = \sum_i w_i \in \mathcal{K}$

- Tai's iterations are not practical because you compute on the finest level in fact
- we added two techniques: *defect obstacles* on each level, and *FAS coarse corrections*

defect obstacles

- recall $\mathcal{K} = \{v \geq \psi\}$ in an obstacle problem

definition

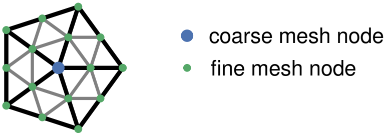
for finest-level admissible set $\mathcal{K}^J = \{v^J \geq \psi^J\} \subset \mathcal{V}^J$ and an iterate $w^J \in \mathcal{K}^J$, the *defect obstacle* (Gräser & Kornhuber 2009) is

$$\chi^J = \psi^J - w^J \in \mathcal{V}^J$$

  - note $\chi^J \leq 0$
- we generate the CD through defect obstacles χ^j on each level via *monotone restriction*:

$$\chi^j = R^{\oplus} \chi^{j+1}$$

  - a *nonlinear* operator
  - also due to (Gräser & Kornhuber 2009)



up and down sets

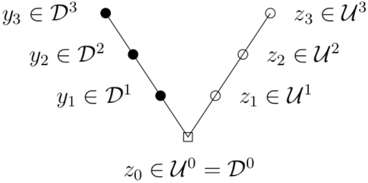
- upward part in the FASCD V-cycle uses large admissible sets:

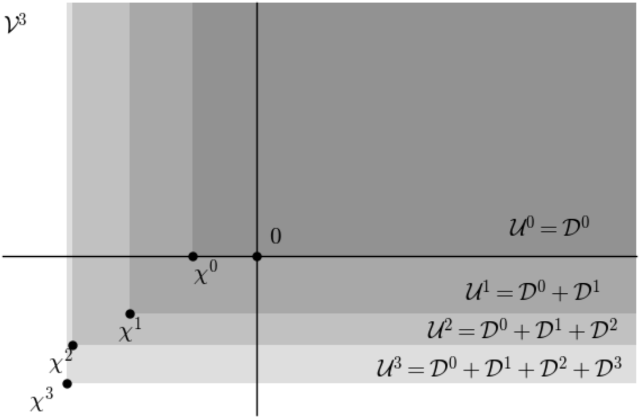
$$\mathcal{U}^j = \{z^j \geq \chi^j\}$$

- downward sets are smaller to guarantee admissibility of the upcoming coarse correction:

$$\mathcal{D}^j = \{y^j \geq \phi^j = \chi^j - \chi^{j-1}\}$$

- $\mathcal{U}^j = \sum_{i=0}^{j} \mathcal{D}^i$ is a CD of the jth-level admissible set

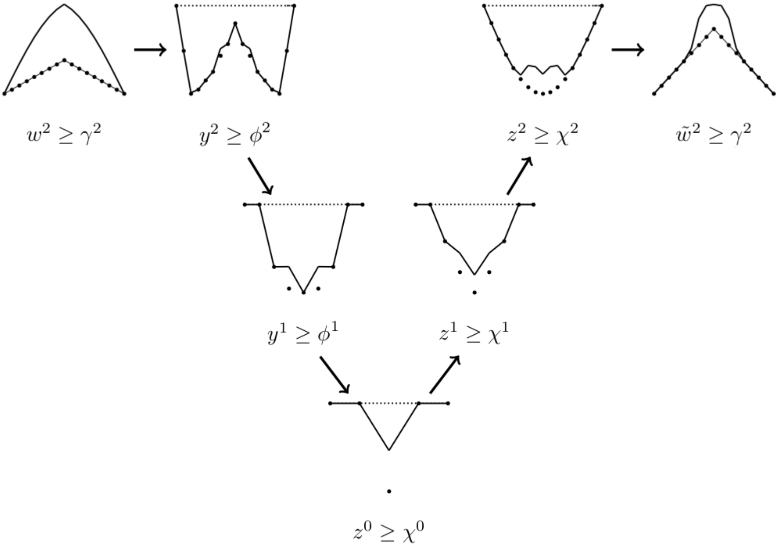



$\mathcal{V}^3$
0
$\mathcal{U}^0 = \mathcal{D}^0$
χ^0
$\mathcal{U}^1 = \mathcal{D}^0 + \mathcal{D}^1$
χ^1
$\mathcal{U}^2 = \mathcal{D}^0 + \mathcal{D}^1 + \mathcal{D}^2$
χ^2
$\mathcal{U}^3 = \mathcal{D}^0 + \mathcal{D}^1 + \mathcal{D}^2 + \mathcal{D}^3$
χ^3

full approximation scheme constraint decomposition

$$
\begin{aligned}
&\text{FASCD-VCYCLE}(J, \ell^J, \psi^J; w^J):\\
&\qquad \chi^J = \psi^J - w^J\\
&\qquad \text{for } j = J \text{ downto } j = 1\\
&\qquad\qquad \chi^{j-1} = R^{\oplus} \chi^j\\
&\qquad\qquad \phi^j = \chi^j - P\chi^{j-1}\\
&\qquad\qquad y^j = 0\\
&\qquad\qquad \text{SMOOTH}^{\mathrm{down}}(\ell^j, \phi^j, w^j; y^j)\\
&\qquad\qquad w^{j-1} = R^{\bullet}(w^j + y^j)\\
&\qquad\qquad \ell^{j-1} = f^{j-1}(w^{j-1}) + R\left(\ell^j - f^j(w^j + y^j)\right)\\
&\qquad z^0 = 0\\
&\qquad \text{SOLVE}(\ell^0, \chi^0, w^0; z^0)\\
&\qquad \text{for } j = 1 \text{ to } j = J\\
&\qquad\qquad z^j = y^j + Pz^{j-1}\\
&\qquad\qquad \text{SMOOTH}^{\mathrm{up}}(\ell^j, \chi^j, w^j; z^j)\\
&\qquad \text{return } w^J + z^J
\end{aligned}
$$

FASCD V-cycle: visualization on a 1D problem



FASCD specifics

see paper (Bueler & Farrell 2023) for:

- generalization to upper and lower obstacles:

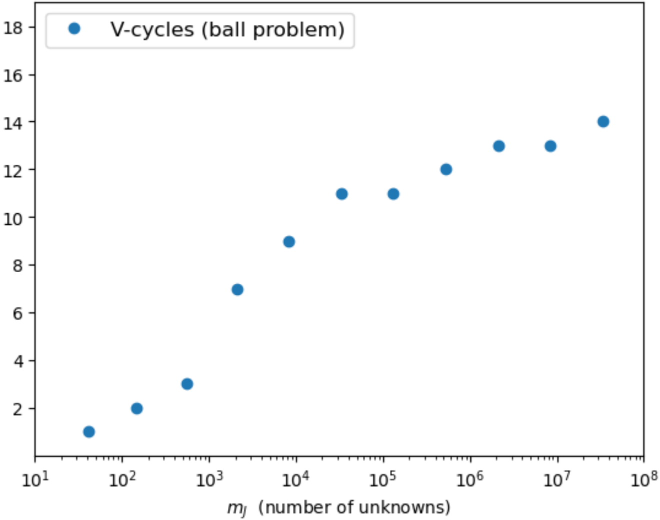
$$\mathcal{K}^J = \{\underline{\psi}^J \le v^J \le \overline{\psi}^J\}$$

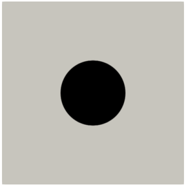
- stopping criteria
    - evaluate whether CP/KKT conditions are satisfied
- FMG cycle
- details of $O(m_J)$ smoother

Outline

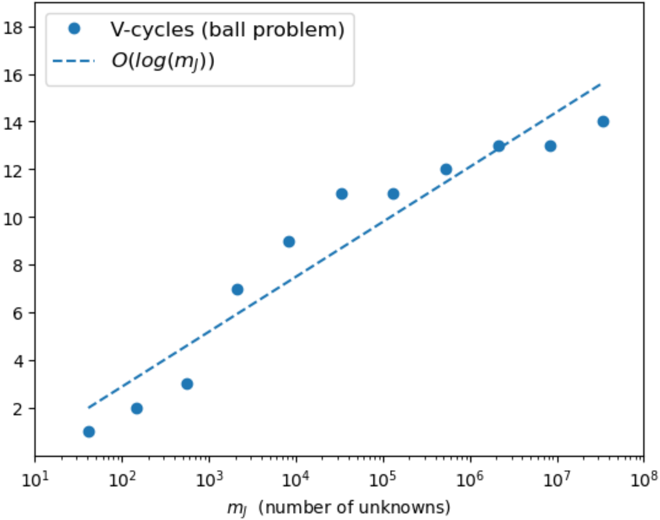
1. variational inequalities (VIs)
2. nonlinear multigrid for PDEs
   - full approximation scheme (FAS)
3. multigrid for VIs
   - full approximation scheme constraint decomposition (FASCD)
4. results
   - classical obstacle problem
   - advection-diffusion of a concentration
   - glacier surface elevations

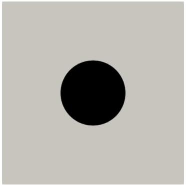
classical obstacle problem by FASCD



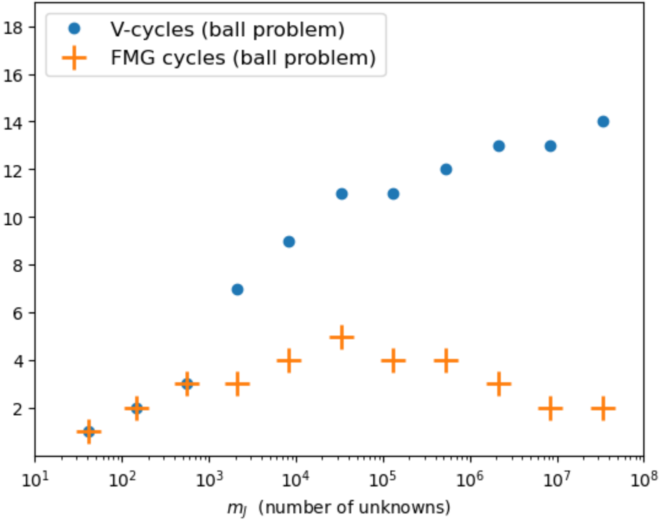


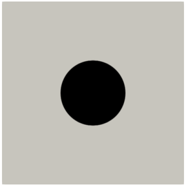
classical obstacle problem by FASCD



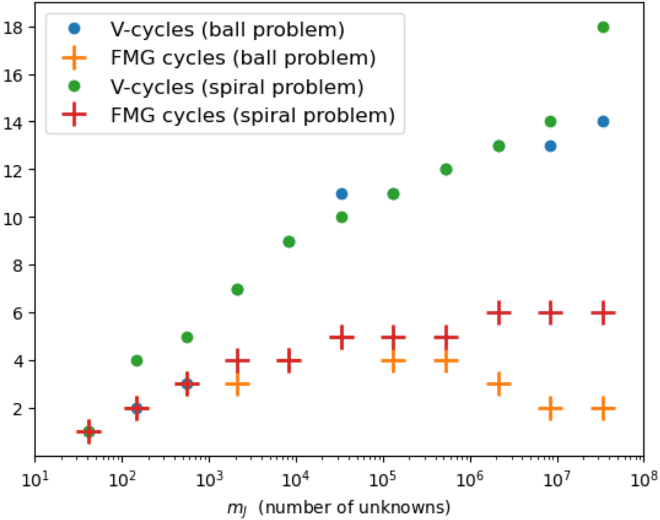


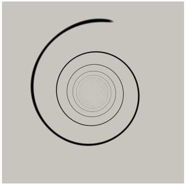
classical obstacle problem by FASCD





classical obstacle problem by FASCD





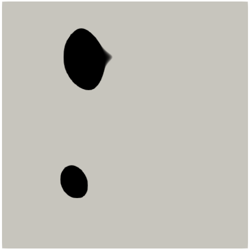
advection-diffusion of a concentration

- suppose $u(x)$ is a concentration in $\Omega \subset \mathbb{R}^d$: $\qquad 0 \le u \le 1$
- suppose it moves by combination of diffusion, advection by wind $\boldsymbol{X}(x)$, and source function $\phi(x)$:

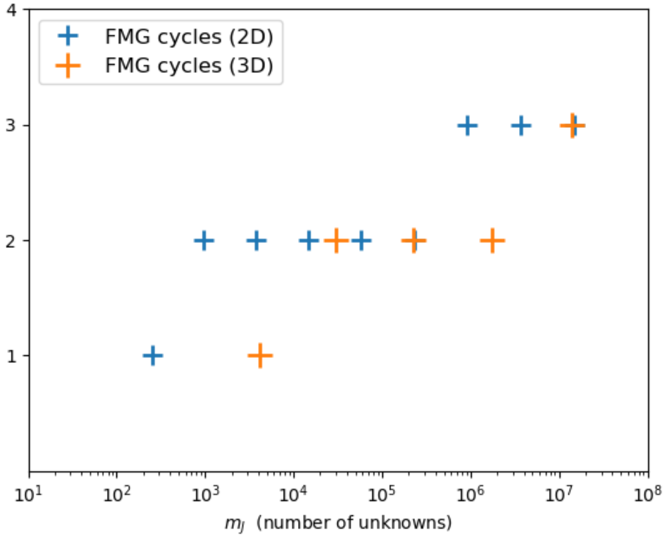
$$-\epsilon\nabla^2 u + \boldsymbol{X} \cdot \nabla u = \phi$$

- two active sets ($d = 2$ *case*):

$$\underline{A}_u = \{u(x) = 0\} \qquad\qquad \overline{A}_u = \{u(x) = 1\}$$

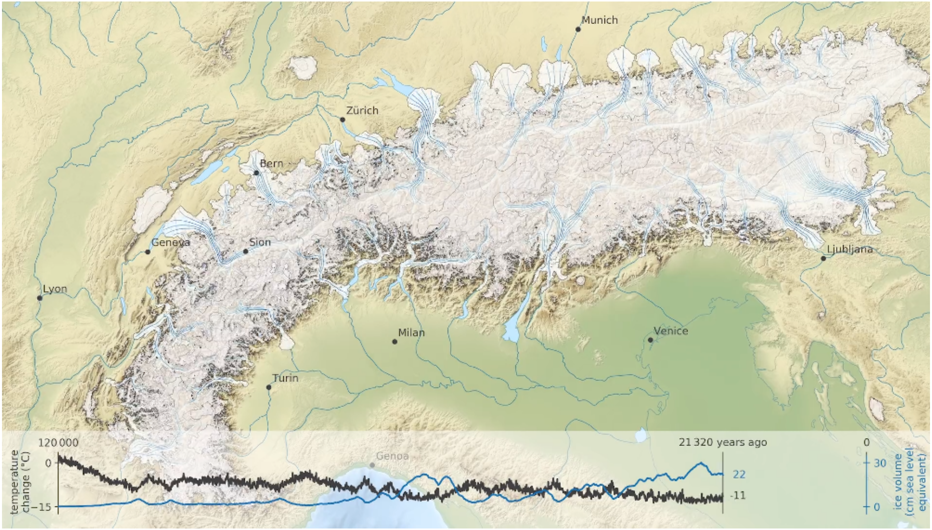


advection-diffusion of a concentration



compare: linear programming (Klee-Minty cube?), spatial correlations

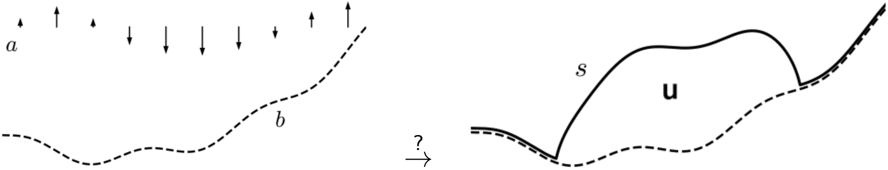
- “where are there glaciers?” is a free-boundary problem



Sequinot et al. (2018)

free-boundary problem: flowing glacier ice in a climate

- glacier = incompressible, viscous fluid driven by gravity
- to find: ice surface elevation $s(t, x, y)$ and velocity $\mathbf{u}(t, x, y, z)$
- over fixed bed topography with elevation $b(x, y)$
  - $s(t, x, y) \geq b(x, y)$
- in a *climate* which adds or removes ice at a signed rate $a(t, x, y)$
  - data a, b is defined on domain $\Omega \subset \mathbb{R}^2$

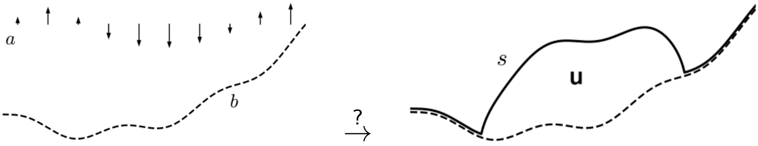


- is this an adequate description?:

$$s \geq b \qquad \text{everywhere in } \Omega$$

$$\frac{\partial s}{\partial t} = a + \mathbf{u}|_s \cdot \mathbf{n}_s \qquad \text{where } s(t,x,y) > b(x,y)$$

- notes:
  - surface velocity $\mathbf{u}|_s$ is, **in some manner**, determined by s
    - $\mathbf{u}|_s$ is generally a *non-local* function of s
  - $\mathbf{n}_s = \langle -\nabla s, 1 \rangle$ is upward surface normal



glacier free-boundary problem: steady VI form

- admissible surface elevations:[1]

$$\mathcal{K} = \{r \in \mathcal{V} \,:\, r \geq b\}$$

- steady ($\frac{\partial s}{\partial t} = 0$) VI problem for surface elevation $s \in \mathcal{K}$:

$$\langle \Phi(s) - a, r - s \rangle \geq 0 \quad \text{for all } r \in \mathcal{K}$$

where

$$\Phi(s) = -\mathbf{u}|_s \cdot \mathbf{n}_s$$

with extension by 0 to all of Ω

[1] $(s-b)^{8/3} \overset{?}{\in} W^{1,4}(\Omega)$ so $\mathcal{V} \overset{?}{=} (W^{1,4})^{3/8}$ ...see (Jouvet & Bueler, 2012)

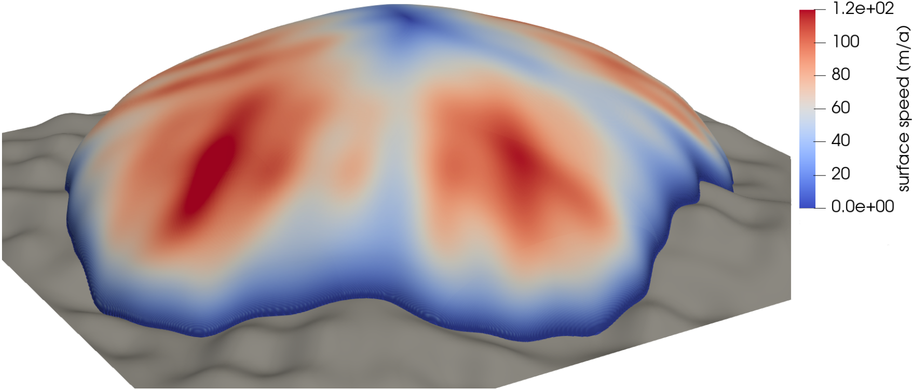
shallow ice approximation

- the *shallow ice approximation* is a highly-simplified view of conservation of momentum
- isothermal, nonsliding case:

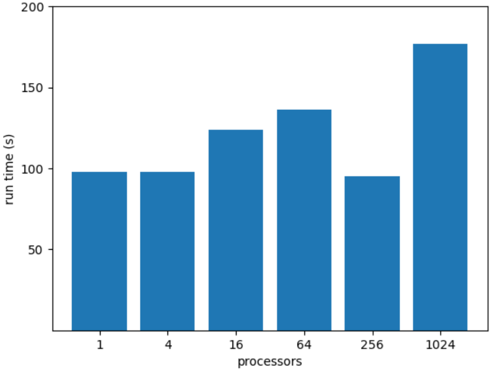
$$\begin{aligned}\Phi(s) &= -\mathbf{u}|_s \cdot \mathbf{n}_s \\ &= -\frac{\gamma}{4}(s-b)^4|\nabla s|^4 - \nabla \cdot \left(\frac{\gamma}{5}(s-b)^5|\nabla s|^2\nabla s\right)\end{aligned}$$

FASCD test case: simplified ice sheet the size of Greenland

- *ice sheet* = big glacier



- observed optimality of FMG solver
- good parallel *weak scaling* as well
  - each processor owns 641×641 (sub) mesh
  - $P = 1024$ run had $20481^2 = 4.1 \times 10^8$ unknowns
    ... and 88 meter resolution



summary and outlook

- FASCD = new multilevel solver for VI (free-boundary) problems
  - implemented in Python Firedrake (over PETSc)
- observed optimality, even TME, in many cases
  - actually fast

to do:

- add mesh adaptivity to free boundary (Stefano)
- implement in C inside PETSc
- apply to space-time (*parabolic*) VI problems
- prove convergence
- identify smoothers for problems like elastic contact
- include membrane stresses in glacier case

references

- A. Brandt (1977). *Multi-level adaptive solutions to boundary-value problems*, Mathematics of Computation 31 (138), 333–390 doi:10.1090/S0025-5718-1977-0431719-X
- E. Bueler & P. Farrell (2023). *A full approximation scheme multilevel method for nonlinear variational inequalities*, submitted arxiv:2308.06888
- C. Gräser & R. Kornhuber (2009). *Multigrid methods for obstacle problems*, J. Comput. Math., 1–44
- G. Jouvet & E. Bueler (2012). *Steady, shallow ice sheets as obstacle problems: well-posedness and finite element approximation*, SIAM J. Appl. Math. 72 (4), 1292–1314 doi:10.1137/110856654
- X. Tai (2003). *Rate of convergence for some constraint decomposition methods for nonlinear variational inequalities*, Numer. Math. 93 (4), 755–786 doi:10.1007/s002110200404
- J. Xu (1992). *Iterative methods by space decomposition and subspace correction*, SIAM Rev. 34 (4), 581–613 doi:10.1137/1034116

background references

- E. Bueler (2021). *PETSc for Partial Differential Equations*, SIAM Press, Philadelphia
- R. Glowinski (1984). *Numerical Methods for Nonlinear Variational Problems*, Springer, Berlin
- N. Kikuchi & J. Oden (1988). *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, SIAM Press, Philadelphia
- D. Kinderlehrer & G. Stampacchia (1980). *An Introduction to Variational Inequalities and their Applications*, Academic Press, New York
- U. Trottenberg, C. Oosterlee, & A. Schuller (2001). *Multigrid*, Elsevier, Oxford