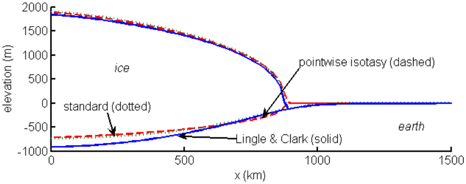
Fast computation of a viscoelastic deformable Earth model for ice sheet simulations

Ed Bueler[1] Craig Lingle[2] Jed Brown[1]

[1]Dept. of Mathematics and Statistics, University of Alaska, Fairbanks

[2]Geophysical Institute, University of Alaska, Fairbanks

International Glaciological Society Symposium on Cryospheric Indicators of Global Climate Change, Aug. 2006



Before I start into it ...

This is a mathematical modelling and numerical methodology talk on one aspect of ice-sheet modelling.

Why might you care?
(And why does the topic of earth deformation matter?)

Three answers:

1. Ice sheet modelling over significant time scales (e.g. ≥ 200 yrs) must include a coupled earth deformation component to predict relative sea level changes near the margin/grounding line of the ice sheet.
2. By *changing the condition at grounding lines* and by *altering the surface slope*, earth deformation effects ice flow.
3. *Fast computation* of earth deformation gives your computer time to do flow simulation (*the harder problem!*).

Outline

Outline

The earth model

- Current standard ELRA model
- Plate over viscous half-space
- Spherical purely-elastic earth

Implementation

Results

- Verification
- Differences among Earth deformation models
- Initialization using current uplift

UNIVERSITY OF ALASKA FAIRBANKS

standard ELRA
= Elastic plate Lithosphere plus Relaxing Asthenosphere

$$\rho_r g w + D\nabla^4 w = \rho_i g H,$$
$$\frac{\partial u}{\partial t} = -\frac{u - w}{\tau}$$

where

w	equilibrium plate position
u	time-dependent plate position
D	flexural rigidity of plate
$\rho_r g w$	bouyant restoring force
$\rho_i g H$	the ice load [H = thickness]
τ	fixed relaxation time (= 3000 yrs)

What's missing?

1. purely-elastic rebound
2. spherical and layered effects
3. where is viscosity of asthenosphere?

Lingle and Clark (1985 JGR) two-component Earth model

- intermediate between simplified models used in current ice sheet simulations and full spherical models used for (serious) earth deformation studies
- implementation based on the Green's functions of two different linear Earth models ("components" here on):
  1. elastic plate lithosphere over viscous half-space (Cathles 1975)
  2. layered purely-elastic spherical self-gravitating Earth (Farrell 1972)
- each component gives displacement, but Earth models are linear, so you can add them:

$$u^{\text{total}} = u^{\text{viscous}} + u^{\text{purely-elastic}}$$

In (Lingle & Clark 1985) the model was applied to a WAIS ice sheet/ice stream flow band and grounding line retreat.

component 1: plate over viscous half-space

Equation from (Lingle & Clark 1985); from close reading of (Cathles 1975):

$$2\eta\kappa\frac{\partial \bar{u}}{\partial t} + \rho_r g \bar{u} + D\kappa^4 \bar{u} = \overline{\rho_i g H}$$

where

$$\eta \qquad = \text{viscosity of asthenosphere}$$

- Hankel transformations! Arrggh!

component 1: plate over viscous half-space

Equation from (Lingle & Clark 1985); from close reading of (Cathles 1975):

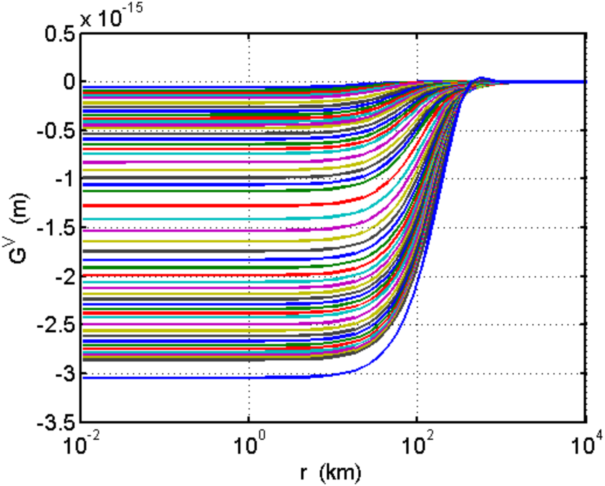
$$2\eta\kappa\frac{\partial \bar{u}}{\partial t} + \rho_r g \bar{u} + D\kappa^4 \bar{u} = \overline{\rho_i g H}$$

where

$$\eta \qquad = \text{viscosity of asthenosphere}$$

- Hankel transformations! Arrggh!
- Note: Hankel transformations are Fourier transforms of two variable functions, but done in polar coordinates.
- What does it predict? Consider a point mass . . .

the Green's function (for illustration; not directly used)



top curve at 20 years, lowest at 100k years

the underlying PDE

a single time-dependent equation for vertical plate displacement:

$$2\eta\,|\nabla|\,\frac{\partial u}{\partial t} + \rho_r g u + D\nabla^4 u \stackrel{\star}{=} \rho_i g H$$

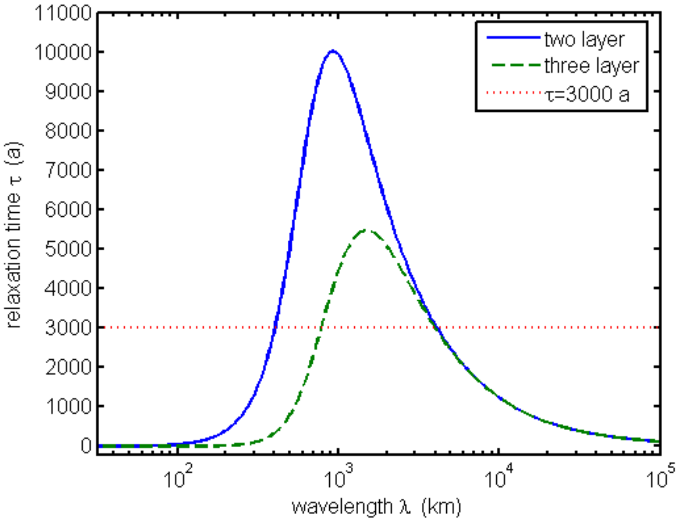
u	plate displacement	g	accel. of grav.
η	viscosity of asthenosphere	D	flexural rigidity
ρ_r	density of asthenosphere	$\rho_i g H$	ice load

[compare ELRA:

$$\rho_r g w + D\nabla^4 w = \rho_i g H, \qquad \frac{\partial u}{\partial t} = -\frac{u - w}{\tau}]$$

$\star$ is what ELRA wants to be!

a relaxation-time spectrum.



what's that "$|\nabla|$" thing?

Definition by the Fourier transform:

$$|\nabla| f = \mathcal{F}_2^{-1}\left[(\xi^2+\zeta^2)^{1/2}\mathcal{F}_2 f\right]$$

where $\mathcal{F}_2$ is the two-variable Fourier transform and ξ, ζ are the Fourier variables. So $|\nabla| = \sqrt{-\nabla^2}$.

Recall that

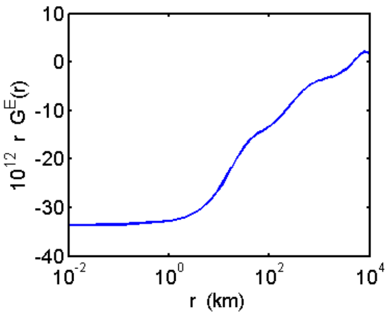
$$\widehat{\frac{\partial \phi}{\partial x}} = i\xi\hat{\phi}(\xi)$$

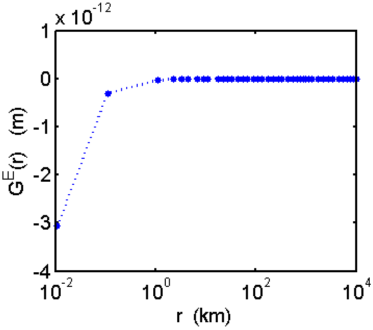
for the one-variable Fourier transform.

Conclusion: $|\nabla|$ is not a differential operator, but very similar. It is just as computable in the Fourier domain.

component 2: purely-elastic, spherical, layered, self-gravitating earth

- Elastic deformations are instantaneous (for ice flow purposes!) and linear.
- Convolution of current load with Green's functions works well.





Outline

Fourier spectral methods

General idea of Fourier approximation methods

Discretize by choosing finitely-many Fourier modes to approximate solution.

Application to PDEs

Derivatives are approximated by simple multiplication in Fourier space.

The method for component 1 (plate over viscous)

Let p, q be the discrete Fourier frequencies (for variables x, y)

Approximate

$$2\eta\,|\nabla|\,\frac{\partial u}{\partial t} + \rho_r g u + D\nabla^4 u = \rho_i g H$$

by

$$2\eta\,\kappa\,\frac{U_{pq}^{n+1} - U_{pq}^{n}}{\Delta t} + \rho_r g U_{pq}^{n+1/2} + D\kappa^4 U_{pq}^{n+1/2} = \rho_i g H_{pq}^{n+1/2}$$

where

$$\kappa = (p^2 + q^2)^{1/2}$$

Step forward in time by iterating n. Note $\Delta t = 100$ yrs is sufficently short in practice

component 2 (purely-elastic spherical) implemented through Green's function

We do it the old-fashioned way:

- Think of the gridded ice load as an array of rectangular blocks.
- For block loads, do high quality numerical integral against Green's function *once at beginning of run*. (I.e. build a load response matrix.)
- For general ice load, do one matrix multiplication per grid point to get Earth surface position.
- When matrix multiplication is implemented as a discrete convolution product by FFT then cost per Earth deformation step is $O(N^2(\log N)^2)$ on an $N \times N$ grid.

Outline

The earth model

- Current standard ELRA model
- Plate over viscous half-space
- Spherical purely-elastic earth

Implementation

Results

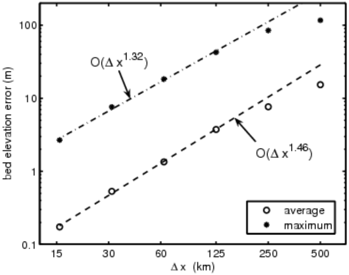
- Verification
- Differences among Earth deformation models
- Initialization using current uplift

Comparison of numerical computation to exact solution

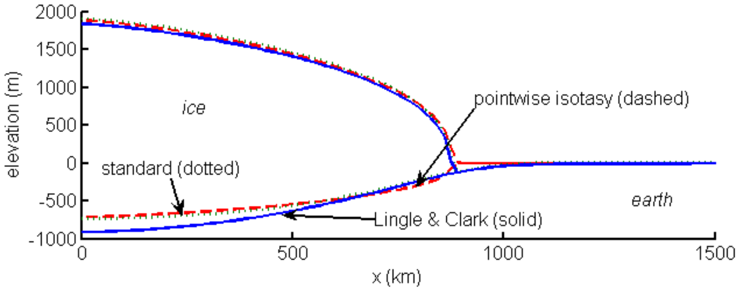
For a disc load, the exact solution to the "plate over viscous half-space" model is known though an integral:

$$u(r,t) = \rho_i g H_0 R_0 \int_0^\infty \beta^{-1} \left\{\exp(-\beta t/(2\eta\kappa)) - 1\right\} J_1(\kappa R_0)\, J_0(\kappa r)\, d\kappa$$

One can therefore report numerical errors made by our implementation of this model:

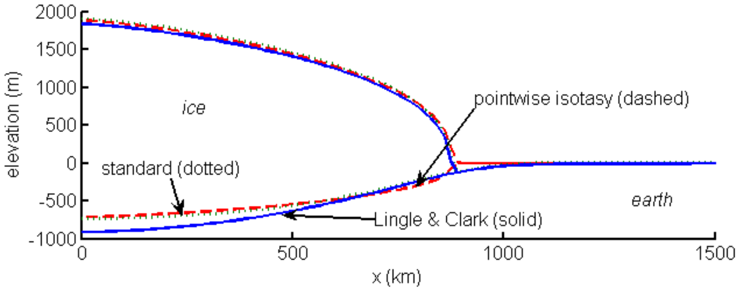


Simplified ice sheet on deforming bed after 60k years



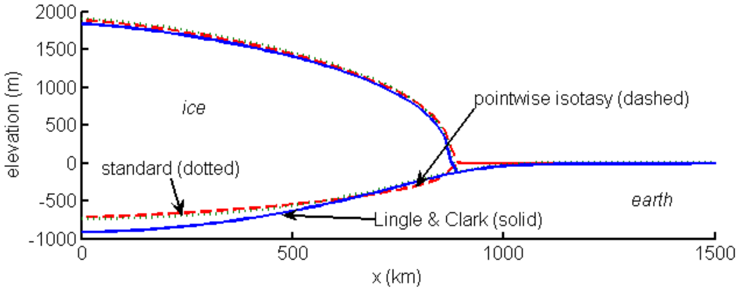
- Shown: Result of coupled isothermal ice sheet flow and three different Earth deformation models.

Simplified ice sheet on deforming bed after 60k years



- Shown: Result of coupled isothermal ice sheet flow and three different Earth deformation models.
- This sheet is an exact continuum solution for pointwise isostasy.

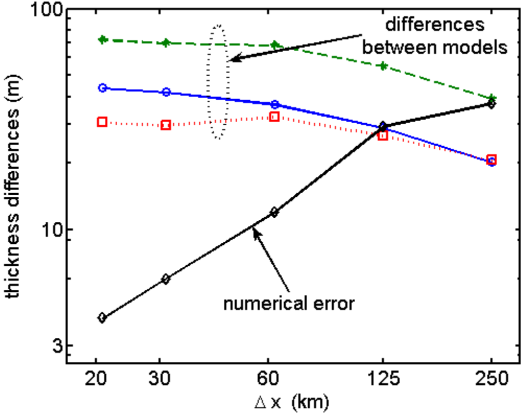
Simplified ice sheet on deforming bed after 60k years



- Shown: Result of coupled isothermal ice sheet flow and three different Earth deformation models.
- This sheet is an exact continuum solution for pointwise isostasy.
- So we know how large the coupled (ice flow)/(earth deformation) numerical error actually is for one of the three models.

Differences among Earth models exceeds numer. error

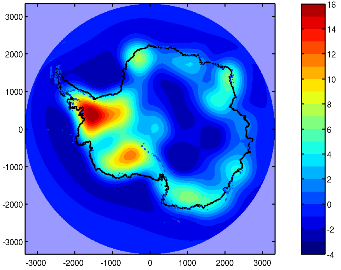
- green = pointwise isostasy *vs* Lingle and Clark
- blue = pointwise isostasy *vs* ELRA
- red = ELRA *vs* Lingle and Clark



There may be current uplift maps available ...

- Current uplift rate can be observed in some locations (e.g. by GPS at exposed bedrock and at raised beaches).
- Add computations with a sophisticated and computationally-expensive Earth model to get a good current uplift map.
- Use to initialize Earth deformation models, at least for 100yr to 10kyr predictions. [prebending idea]

Example: Uplift for Antarctica in m/yr (Ivins and James 1998)



Initialization of by an uplift rate map: How-to

Solve this equation for initial plate displacement u_0 given load and uplift rate map:

$$\rho_r g u_0 + D\nabla^4 u_0 = \rho_i g H_0 - 2\eta\,|\nabla|\,(\text{UPLIFT RATE})$$

where

$\rho_i g H_0$	current ice load
UPLIFT RATE	map-plane function $\left.(\partial u/\partial t)\right\|_{t=0}$.

Removes need for equilibrium assumption at time zero.

Summary

the Lingle and Clark (1985) Earth model

- combines a layered self-gravitating elastic spherical Earth
- *and* a plate lithosphere over viscous half-space
- has geophysically reasonable relaxation spectrum (generalizes ELRA)
- "adds back" purely-elastic deformation missing from ELRA

the new FFT implementation

- Fast! $O(N^2(\log N)^2)$ per time step on an $N \times N$ grid

Extra: Adding another asthenosphere layer (e.g. a low viscosity channel)

$$2\eta \left|\nabla\right| R\left(\left|\nabla\right|\right) \frac{\partial u}{\partial t} + \rho_r g u + D\nabla^4 u = \sigma_{zz}$$

where

$$R(\kappa) = \frac{2\tilde{\eta}C(\kappa)S(\kappa) + (1-\tilde{\eta}^2)T_c^2\kappa^2 + \tilde{\eta}^2 S(\kappa)^2 + C(\kappa)^2}{(\tilde{\eta}+\tilde{\eta}^{-1})C(\kappa)S(\kappa) + (\tilde{\eta}-\tilde{\eta}^{-1})T_c\kappa + S(\kappa)^2 + C(\kappa)^2}$$